Equations différentielles stochastiques rétrogrades ergodiques et applications aux EDP

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Introduction

La première partie de cette introduction est constituée de rappels sur les EDSR en dimension finie et infinie. La deuxième et la troisième partie sont constituées de rappels sur les thèmes abordés et présentent les résultats obtenus lors de cette thèse.

1.1 Rappels préliminaires sur la théorie des EDSR

1.1.1 Justification de la structure des EDSR

Donnons nous un mouvement Brownien W, k-dimensionnel défini sur un espace probabilisé complet $(\Omega, \mathscr{F}, \mathbb{P})$ dont la filtration naturelle augmentée est notée $(\mathscr{F}_t)_{t\in[0,T]}$. Imaginons à présent que l'on souhaite résoudre l'équation différentielle suivante :

$$\begin{cases} \mathrm{d}Y_t = -f(t, Y_t)\mathrm{d}t, \quad \forall t \in [0,T], \\ Y_T = \xi, \end{cases} \tag{1.1}$$

où ξ est une variable aléatoire $\mathscr{F}_T$-mesurable, c'est-à-dire une variable aléatoire connue à l'instant T. Le temps T est aussi parfois appelé horizon. Supposons pour simplifier que $f \equiv 0$, le problème (1.1) devient alors

$$\begin{cases} \mathrm{d}Y_t = 0 \quad \forall t \in [0,T], \\ Y_T = \xi. \end{cases} \tag{1.2}$$

Un candidat solution à ce problème est alors $Y_t = \xi$, $\forall t \in [0,T]$. Cependant, si nous demandons à la solution de ne pas dépendre du futur, c'est-à-dire d'être adapté à la filtration générée par le mouvement brownien $(\mathscr{F}_t)_{t\in[0,T]}$, alors la solution proposée ne convient pas. Un moyen naturel de rendre adapté ξ sans changer sa valeur terminale est de considérer son espérance conditionnelle par rapport à la filtration du mouvement brownien. Un nouveau candidat solution est alors $Y_t = \mathbb{E}(\xi|\mathscr{F}_t)$. Comme ce terme n'est a priori pas différentiable en temps au sens usuel, nous utilisons le théorème de représentation des martingales browniennes pour faire apparaître une intégrale stochastique. Y_t étant une martingale brownienne, il existe un processus Z adapté et de carré intégrable tel que, pour tout $t \in [0,T]$,

$$Y_t = \mathbb{E}(\xi) + \int_0^t Z_s \mathrm{d}W_s.$$

En différenciant la relation précédente il apparaît que $Y_t = \mathbb{E}(\xi|\mathscr{F}_t)$ résout l'équation suivante :

$$\begin{cases} \mathrm{d}Y_t = Z_t \mathrm{d}W_t \quad \forall t \in [0,T], \\ y_T = \xi. \end{cases} \tag{1.3}$$

Manifestement, la structure de l'équation initiale (1.2) a été modifiée, faisant apparaître un nouveau terme $Z_t\mathrm{d}W_t$ qui permet de rendre adaptée la solution. Revenons à présent au problème initial (1.1), comme nous introduisons un terme supplémentaire Z dans l'équation, il est naturel d'autoriser la fonction f à dépendre de Z, ce qui nous conduit au problème :

$$\begin{cases} \mathrm{d}Y_t = -f(t, Y_t, Z_t)\mathrm{d}t + Z_t\mathrm{d}W_t, & \forall t \in [0, T], \\ Y_T = \xi, \end{cases} \tag{1.4}$$

ou encore en utilisant la formulation intégrale rétrograde faisant apparaître la condition terminale :

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\mathrm{d}s - \int_t^T Z_s\mathrm{d}W_s, \quad 0 \le t \le T, \tag{1.5}$$

Les données de cette équation sont, d'une part la condition terminale ξ qui est une variable aléatoire $\mathscr{F}_T$-mesurable à valeur dans $\mathbb{R}^m$ et d'autre part le générateur f qui est une fonction $\Omega \times [0, T] \times \mathbb{R}^m \times \mathbb{R}^{m\times k} \to \mathbb{R}^m$ mesurable par rapport aux tribus $\mathscr{P} \otimes \mathscr{B}(\mathbb{R}^m) \otimes \mathscr{B}(\mathbb{R}^{m\times k})$ et $\mathscr{B}(\mathbb{R}^m)$, où $\mathscr{P}$ est la tribu des événements prévisibles. L'inconnu d'une telle équation est le couple de processus $(Y_t, Z_t)_{t\in[0,T]}$ à valeur dans $\mathbb{R}^m \times \mathbb{R}^{m\times k}$.

Définition 1.1. *Une solution de l'EDSR* (1.5) *est un couple de processus* $(Y_t, Z_t)_{t\in[0,T]}$ *à valeur dans* $\mathbb{R}^m \times \mathbb{R}^{m\times k}$ *tel que*

1. $(Y_t)_{t\in[0,T]}$ *est à trajectoires continues* $\mathbb{P}$*-p.s. et adapté,* $(Z_t)_{t\in[0,T}$ *est prévisible,*
2. $$\int_0^T \left[|f(s, Y_s, Z_s)| + |Z_s|^2\right] \mathrm{d}s < +\infty, \quad \mathbb{P}\text{-}p.s.,$$
3. $$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\mathrm{d}s - \int_t^T Z_s\mathrm{d}W_s, \quad 0 \le t \le T, \quad \mathbb{P}\text{-}p.s.$$

1.1.2 Un résultat fondateur

Les EDSRs apparaissent pour la première fois dans un article de Bismut [9] dans le cas où le générateur est linéaire. Cependant le point de départ de la théorie des EDSRs est l'article de Pardoux et Peng [68] dans lequel le générateur est non linéaire par rapport à y et z. Rappelons ce résultat.

Théorème 1.2 (Pardoux-Peng 1990)**.** *Supposons le générateur f lipschitzien par rapport à (y, z), uniformément en (t, w), et*

$$\mathbb{E}\left[|\xi|^2 + \int_0^T |f(s, 0, 0)|^2\mathrm{d}s\right] < +\infty.$$

Alors l'EDSR (1.5) admet une unique solution telle que Z soit un processus de carré intégrable.

Remarquons que les hypothèses sur f sont très proches de celles que contient le théorème de Cauchy-Lipschitz pour les EDO ($f(t, y)$ continue en les deux variables et Lipschitz par rapport à la seconde variable pour l'existence et l'unicité d'une solution globale).

1.1.3 Ramifications et terminologie

A partir de cette étude, plusieurs chemins d'investigations peuvent être envisagés. Nous n'avons pas l'ambition d'en dresser une liste exhaustive dans ce manuscrit. Commençons par le cas des EDSRs dont le générateur n'est plus Lipschitz en y mais seulement monotone au sens suivant : $\exists \mu \in \mathbb{R}$, $\forall t \in \mathbb{R}_+, \forall z \in \mathbb{R}^{m\times k}, \forall y, y' \in \mathbb{R}^m$,

$$\langle y - y', f(t,y,z) - f(t,y',z)\rangle \leq \mu |y-y'|^2.$$

Ce type d'hypothèse apparaît pour la première fois dans un article de Peng [71] pour traiter le cas d'une EDSR où l'horizon est un temps aléatoire, c'est-à-dire en imposant $Y_\tau = \xi$ où τ est un temps d'arrêt, autrement dit :

$$Y_t = \xi + \int_{t\wedge\tau}^{\tau} f(s, Y_s, Z_s)\mathrm{d}s - \int_{t\wedge\tau}^{\tau} Z_s \mathrm{d}W_s, \qquad 0 \leq t \leq T. \tag{1.6}$$

Une telle EDSR est appelée EDSR en horizon aléatoire. Rappelons le résultat de Darling et Pardoux [24] à ce sujet. Lorsque la condition de monotonie ci-dessus est vérifiée et que le générateur est Lipschitz en z, à croissance linéaire en y, z (c'est-à-dire qu'il existe un processus f_t positif tel que $\forall t \in \mathbb{R}_+, \forall y \in \mathbb{R}^m, \forall z \in \mathbb{R}^{m\times k}$, $|f(t,y,z)| \leq f_t + C|y| + K|z|$), que le générateur est continu en y à t et z fixés, et que la condition d'intégrabilité suivante est vérifiée : $\mathbb{E}\left[|\xi|^2 + \int_0^T f_t^2 \mathrm{d}t\right] < +\infty$, alors l'EDSR (1.5) admet une unique solution telle que Z est un processus de carré intégrable. En prenant une condition terminale nulle et $\tau = +\infty$, on obtient l'EDSR suivante :

$$Y_t = \int_t^{+\infty} f(s, Y_s, Z_s)\mathrm{d}s - \int_t^{+\infty} Z_s \mathrm{d}W_s, \qquad \forall t \geq 0, \tag{1.7}$$

ou encore, par soustraction, $\forall T \geq 0$,

$$Y_t = Y_T + \int_t^T f(s, Y_s, Z_s)\mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s, \qquad \forall t \in [0,T].$$

Ce type d'EDSR porte le nom d'EDSR en horizon infini. Un autre cas très largement étudié est celui des EDSRs markoviennes, dans lequel le générateur et la condition terminale ne dépendent de l'aléa w qu'au travers d'un processus solution d'une EDS. Plus précisément, considérons l'EDS partant de x au temps t :

$$X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x})\mathrm{d}r + \int_t^s \sigma(r, X_r^{t,x})\mathrm{d}W_r, \qquad \forall s \in [t,T]. \tag{1.8}$$

Sous de bonnes hypothèses, l'EDS précédente admet une unique solution forte. L'EDSR markovienne s'écrit alors :

$$Y_s^{t,x} = g(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})\mathrm{d}r - \int_s^T Z_r^{t,x}\mathrm{d}W_r, \qquad \forall s \in [t,T], \tag{1.9}$$

où g est une fonction déterministe. Nous constatons que les coefficients de l'EDS précédente ne dépendent pas de $(Y_s^{t,x}, Z_s^{t,x})_{s\in[t,T]}$, c'est pourquoi l'on qualifie le système formé par les équations (1.8) et (1.9) de découplé. Réciproquement, lorsque nous autorisons les coefficients de l'EDS à dépendre de $(Y_s^{t,x}, Z_s^{t,x})_{s\in[t,T]}$, nous parlons de système couplé. Nous renvoyons le lecteur à l'article de Pardoux Peng [67] pour une étude du cas découplé et à l'article de Ma, Protter et Yong [53] pour une étude du système couplé. Un autre cas intéressant est celui des EDSR quadratiques, où le générateur n'est plus Lipschitz en z mais, comme son nom l'indique, à croissance quadratique en z. Plus précisément, le générateur doit vérifier :

$$\forall t \in \mathbb{R}_+, y \in \mathbb{R}^m, z \in \mathbb{R}^k, |f(t,y,z)| \leq \alpha_t + \beta|y| + \gamma|z|^2, \qquad \mathbb{P}\text{-p.s.},$$

où $\beta, \gamma \in \mathbb{R}_+$ et α est un processus adapté positif vérifiant la condition d'intégrabilité suivante :

$$\exists C > 0, \int_0^T \alpha_s \mathrm{d}s \leq C, \qquad \mathbb{P}\text{-p.s.}$$

Nous renvoyons le lecteur à l'article fondateur de Kobylanski [48] pour plus de précisions sur le sujet. Enfin citons un dernier cas d'étude des EDSRs, il s'agit des EDSRs généralisées qui font apparaître un terme intégral de type Stieltjes dans la structure de l'EDSR. Plus précisément, si $g : \mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{R}^m$ et si $(A_s)_{s\in[0,T]}$ désigne un processus positif croissant, alors l'EDSR

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\mathrm{d}s + \int_t^T g(s, Y_s)\mathrm{d}A_s - \int_t^T Z_s \mathrm{d}W_s, \qquad \forall t \in [0, T],$$

est une EDSR généralisée. Nous renvoyons le lecteur à l'article de Pardoux et Zhang [69] pour des résultats concernant ce type d'EDSR.

1.1.4 Motivation

Depuis le résultat de Pardoux et Peng [68], la théorie des EDSR s'est considérablement développée en raison du lien existant avec les EDPs et des applications possibles aux problèmes de contrôles stochastiques et aux problèmes de mathématiques financières.

Formule de Feynman-Kac

Commençons par évoquer le lien avec les EDPs. Rappelons tout d'abord que les EDS sont reliées aux EDPs par la formule de Feynman-Kac qui donne une solution probabiliste à l'EDP linéaire suivante :

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) + \mathscr{L}u(t,x) - K(t,x)u(t,x) = 0, \qquad \forall t \in [0,T], x \in \mathbb{R}^d, \\ u(T,x) = g(x), \end{cases} \tag{1.10}$$

où $\mathscr{L}$ est un opérateur différentiel du second ordre agissant sur les fonctions h assez régulières par la relation suivante :

$$\mathscr{L}h(t,x) := \frac{1}{2}\operatorname{Tr}\left(\sigma(t,x)^t\sigma(t,x)\nabla^2 h(t,x)\right) + {}^t b(t,x)\nabla h(t,x),$$

où $b : \mathbb{R}^d \to \mathbb{R}^d$ et $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times k}$. Considérons la diffusion de générateur $\mathscr{L}$, c'est-à-dire :

$$X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x})\mathrm{d}r + \int_t^s \sigma(r, X_r^{t,x})\mathrm{d}W_r. \tag{1.11}$$

Alors, sous de bonnes hypothèses pour les coefficients $b, \sigma, K,$ et ψ, il est possible de montrer que la fonction :

$$u(t,x) := \mathbb{E}\left[e^{-\int_t^T K(r, X_s^{t,x})\mathrm{d}s} g(X_T^{t,x})\right]$$

est solution classique (ou solution de viscosité - voir [3] pour une définition - dans le cas de coefficients moins réguliers) de l'EDP (1.10). La théorie des EDSR permet de généraliser ce résultat au cadre des EDP semi-linéaires du second ordre de la forme suivante

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) + \mathscr{L}u(t,x) + f(t,x,u(t,x), {}^t\nabla u\sigma(t,x)) = 0 \qquad \forall t \in [0,T], \forall x \in \mathbb{R}^d, \\ u(T,x) = g(x), \qquad \forall x \in \mathbb{R}^d, \end{cases} \tag{1.12}$$

appelées équations de Kolmogorov. En effet en conservant la même diffusion (1.11) et en considérant l'EDSR suivante :

$$Y_s = g(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})\mathrm{d}r - \int_s^T Z_r^{t,x}\mathrm{d}W_r, \qquad \forall s \in [t,T], \tag{1.13}$$

il est possible de montrer, sous de bonnes hypothèses pour f et g, que $u(t,x) := Y_t^{t,x}$ est une quantité déterministe et est solution classique ou de viscosité suivant la régularité des coefficients de (1.12). Réciproquement on peut montrer par une formule d'Itô appliqué à $v(X_s^{t,x})$, que si $v(t,x)$ est une solution classique alors $(v(s,X_s^{t,x}), {}^t\nabla v(s,X_s^{t,x})\sigma(s,X_s^{t,x}))$ est solution de l'EDSR (1.13). L'unicité des solutions de l'EDSR (1.13) n'implique pas en général l'unicité des solutions de viscosité du problème (1.12). En revanche l'unicité des solutions de viscosité implique l'unicité des solutions markoviennes de l'EDSR. En effet si $(Y_s^1 = u^1(s,X_s^{t,x}), Z_s^1)$ et $(Y_s^2 = u^2(s,X_s^{t,x}), Z_s^2)$ sont deux solutions de l'EDSR (1.13) alors u^1 et u^2 sont deux solutions de viscosité de (1.12) et donc $u^1 = u^2$ par unicité des solutions de viscosité. Il suffit alors de calculer $|Y_T^1 - Y_T^2|^2$ par une formule d'Itô pour obtenir $\mathbb{E}\int_0^T |Z_r^1 - Z_r^2|^2\mathrm{d}r = 0$.

L'intérêt des formules du type Feynman-Kac est de pouvoir utiliser les résultats sur les EDSR à l'étude des EDP ou faire le contraire. Pour plus d'information concernant cette problématique nous renvoyons le lecteur à [67] pour le cadre initial, à [24] pour le cas des EDSR avec horizon aléatoire qui correspond aux EDP semi-linéaires avec condition de Dirichlet au bord, ou à [69] pour les cas des EDP avec condition de Neumann au bord. Enfin pour une approche déterministe des EDP du second ordre (possiblement non-linéaires), nous renvoyons le lecteur à [19] ou [46] pour une approche générale (pas de condition au bord, conditions de Dirichlet ou de Neumann au bord) ou à [2] pour le cas des EDP avec conditions de Neumann au bord. Enfin rappelons que le bon outil pour étudier les EDP non-linéaires par une approche probabiliste est celui des EDSR du second ordre, encore appelées 2EDSR et qui ont été introduites dans [15].

Application aux problèmes de contrôles stochastiques

Les problèmes de contrôle stochastique consistent généralement à étudier la possibilité de minimiser le coût d'un processus solution d'une EDS. La théorie des EDSR permet de traiter efficacement ce genre de problème. Considérons par exemple une diffusion X^u solution d'une EDS contrôlée par un processus u à valeur dans U, espace métrique séparable,

$$X_t^u = x + \int_0^t [b(s,X_s^x) + R(s,X_s^u,u_s)]\mathrm{d}s + \int_0^t \sigma(s,X_s^u)\mathrm{d}W_s, \qquad \forall t \in [0,T]. \tag{1.14}$$

On considère alors un coût qui prend en compte toute la trajectoire et la valeur terminale de X^u :

$$J(u) = \mathbb{E}\left[\int_0^T h(s,X_s^u,u_s)\mathrm{d}s + g(X_T^u)\right].$$

Alors, si σ est inversible, nous pouvons définir l'hamiltonien associé à ce problème de contrôle stochastique par

$$f(t,x,z) = \inf_{u\in U}\left\{h(t,x,u) + z\sigma^{-1}(t,x)R(s,x,u)\right\}.$$

Alors en étudiant l'EDSR dont le générateur est cet hamiltonien et dont la condition terminale est g, $\forall s \in [0,T]$,

$$Y_t = g(X_T) + \int_t^T f(s,X_s,Z_s)\mathrm{d}s - \int_t^T Z_s\mathrm{d}W_s,$$

où X est solution de l'EDS (1.14) avec $R = 0$, il est possible de montrer que pour tout contrôle admissible u, $J(u) \geq Y_0$. Sous certaines hypothèses supplémentaires, il est même possible de montrer que l'égalité est vérifiée pour un contrôle optimal $\overline{u}$ qui est fonction déterministe de X et Z (caractérisation du contrôle optimal par feedback law). De nombreux articles s'aident des EDSR afin de résoudre les problèmes de contrôle optimal, nous renvoyons le lecteur à [10], à [30] pour des problèmes de contrôles stochastiques faisant intervenir des EDSR quadratiques.

Application aux mathématiques financières

Les EDSR permettent de modéliser efficacement certains problèmes de mathématiques financières. N'ayant pas travaillé sur ce sujet, nous renvoyons le lecteur à [29] pour une introduction à cette problématique.

1.1.5 Le cas de la dimension infinie

Processus de Wiener, intégrale stochastique dans les espaces de Hilbert et bruit blanc en espace-temps

L'étude des EDS en dimension infinie (dans les espaces de Banach ou de Hilbert) sont des généralisations naturelles des EDS d'Itô en dimension finie (voir par exemple [47] pour une introduction). Commençons par présenter certains concepts propres au cadre de la dimension infinie.

Soient $(H, \langle\cdot,\cdot\rangle)$ et $(K, \langle\cdot,\cdot\rangle)$ deux espaces de Hilbert séparables. Soit $\{e_i\} \subset H$ un système orthonormé complet de H. Un opérateur linéaire borné $T \in \mathscr{L}(H, K)$ est de Hilbert-Schmidt si

$$\sum_{i\geq 1} ||Te_i||_K^2 < +\infty.$$

Il est bien connu qu'une telle somme est indépendante de la base choisie (voir par exemple [22]). L'ensemble de tous les opérateurs de Hilbert-Schmidt muni de la norme

$$||T||_{\mathscr{L}^2(H,K)} = \left(\sum_{i\geq 1} ||Te_i||_K^2\right)^{1/2},$$

est un espace de Hilbert, noté $\mathscr{L}^2(H, K)$. Il est alors possible d'étendre la définition du mouvement brownien au cadre de la dimension infinie.

Définition 1.3 (Processus de Wiener)**.** *Supposons que $H \subset K$ avec injection Hilbert-Schmidt (i.e. il existe $T : H \to K$ opérateur injectif de Hilbert-Schmidt). On appelle processus de Wiener cylindrique sur H à valeur dans K un processus stochastique W défini sur $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ tel que*

1. *$\forall i \in \mathbb{N}^*$, $t \mapsto \langle W_t, e_i\rangle$ est un mouvement brownien scalaire adapté à la filtration $(\mathscr{F}_t)_{t\geq 0}$;*
2. *$(\langle W, e_i\rangle)_{i\in\mathbb{N}^*}$ est une famille de processus mutuellement indépendants.*

On pose alors,

$$\beta_i(t) = \langle W(t), e_i\rangle \quad \text{et} \quad W_t = \sum_{i\in\mathbb{N}} \beta_i(t)e_i.$$

La série précédente converge dans $L^2(\Omega; U)$ via l'injection T (i.e. $\sum_i \beta_i(t)Te_i$ converge dans $L^2(\Omega; U)$) mais W_t n'est $\mathbb{P}$-p.s. pas à valeurs dans H.

Autrement dit la série précédente ne converge pas dans H, elle ne converge que vue au travers d'une injection qui permet de faire converger la somme en atténuant certains modes. On peut aussi caractériser un tel processus gaussien par

$$\mathbb{E}W_t = 0 \quad \text{dans H,} \quad \mathbb{E}\left(\langle W_s, u\rangle\langle W_t, v\rangle\right) = \min(s,t)\langle u, v\rangle_H$$

pour tout $u, v \in H$. Soit Φ un processus prévisible tel que pour tout $t \in [0,T]$, $\Phi(t)$ soit à valeurs dans $\mathscr{L}^2(H,K)$ et tel que $\int_0^T |\Phi(s)|^2_{\mathscr{L}^2(H,K)}\mathrm{d}s < +\infty$ presque sûrement. Alors on peut définir l'intégrale stochastique $\int_0^T \Phi(s)\mathrm{d}W_s$. La construction se fait similairement au cas de l'intégrale stochastique par rapport à un mouvement brownien. De plus, $\int_0^T \Phi(s)\mathrm{d}W_s$ est une variable aléatoire à valeurs dans K et

$$\int_0^T \Phi(s)\mathrm{d}W_s = \sum_{i=1}^{+\infty}\int_0^T \Phi(s)e_i\mathrm{d}\beta_i(s).$$

Enfin, si $\mathbb{E}\left(\int_0^T ||\Phi(s)||^2_{\mathscr{L}^2(H,K)}\mathrm{d}s\right) < +\infty$, alors

$$\begin{aligned}
&\mathbb{E}\int_0^T \Phi(s)\mathrm{d}W_s = 0,\\
&\mathbb{E}\left(\left|\int_0^T \Phi(s)\mathrm{d}W_s\right|^2\right) = \mathbb{E}\left(\int_0^T ||\Phi(s)||^2_{\mathscr{L}^2(H,K)}\mathrm{d}s\right).
\end{aligned}$$

Concluons cette partie par l'introduction d'un objet qui sera utile dans ce qui suit. On appelle bruit blanc en espace-temps dans $\mathbb{R}_+ \times \mathbb{R}^d$, un processus stochastique $(\eta_t)_{t\in[0,T]}$ gaussien centré à valeurs mesures tel que pour tout Borélien A et B de $\mathbb{R}^d$ et pour tout $s, t > 0$, on a

$$\mathbb{E}\left(\eta_s(A)\eta_t(B)\right) = \lambda_d(A\cap B)\delta_{s-t},$$

où λ désigne la mesure de Lebesgue sur $\mathbb{R}^d$. On parle aussi de processus indexé par la mesure de Lebesgue. Soit W un processus de Wiener cylindrique défini plus haut, on montre que, au sens des distributions,

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \eta,$$

où la dérivée est à comprendre au sens des distributions. Nous renvoyons à [74] pour la justification de ce résultat.

Motivation

Examinons sur un exemple l'intérêt d'étudier les EDS en dimension infinie. Considérons l'équation de la chaleur en dimension un (voir le livre de Cannon [13] pour une étude approfondie de l'équation de la chaleur en dimension un) sur une barre de longueur π et perturbée par un bruit blanc en espace et en temps $\eta(t,x)$, $\forall t \in [0,+\infty)$, $\forall x \in [0,\pi]$,

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u}{\partial x^2}(t,x) + f(x, u(t,x)) + \eta(t,x),\\ u(0,x) = \varphi(x). \end{cases} \tag{1.15}$$

La quantité $u(t,x)$ modélise la température de la barre à l'instant t et à l'emplacement x de la barre. Le terme $\frac{\partial u}{\partial t}$ décrit l'évolution de temporelle de la température. Le terme $\frac{\partial^2 u}{\partial x^2}(t,x)$ traduit le comportement qu'a la chaleur à se déplacer des zones chaudes vers les zones froides. Un point où $\frac{\partial^2 u}{\partial x^2}(t,x) > 0$ est un point plus froid que son entourage direct et dont la température va augmenter (et inversement). Il s'agit donc d'un terme de moyennisation qui va avoir tendance à régulariser les solutions. Le terme non-linéaire $f(x, u(x))$ peut décrire une création interne de chaleur. Le terme φ est le profil de température à l'instant initial. Enfin, $\eta(t,x)$ désigne un bruit blanc en espace et en temps qui vient perturber la répartition de température. Rappelons que, au sens des distributions, $\mathrm{d}W_t = \eta \mathrm{d}t$. Dans la suite de ce paragraphe, nous utiliserons les notations suivantes

- $L^2(0,\pi)$ est l'espace de Hilbert constitué des fonctions dont le carré est Lebesgue intégrable sur $(0,\pi)$, muni du produit scalaire usuel.
- $H^p(0,\pi)$, avec $p \in \mathbb{N}$ est l'espace de Sobolev d'ordre p, c'est-à-dire $H^p(0,\pi) = \left\{h(\cdot) \in L^2(0,\pi) \ : \ \frac{\mathrm{d}^k h}{\mathrm{d}x^k}(\cdot) \in L^2(0,\pi), k = 1, 2, \ldots, p\right\}$ muni du produit scalaire usuel. Rappelons que les dérivées ci-dessus sont à comprendre au sens des distributions et que $H^p(0,\pi)$ muni du produit scalaire usuel (voir par exemple Aubin [1] pour plus de précisions sur les espaces de Sobolev) est un espace de Hilbert.

Définissons un opérateur A par

$$A = \frac{\mathrm{d}^2}{\mathrm{d}x^2}, \text{sur } L^2(0,\pi).$$

Notons

$$H = L^2(0,\pi)$$

l'espace d'état et notons

$$D(A) = H^2(0,\pi) \subset H$$

le domaine de A, alors A est un opérateur (possiblement non borné) de $D(A)$ dans H. Définissons également

$$U(t) = u(t,\cdot),$$
$$F(y)(x) = f(x, y(x)),$$

Alors, il est possible de regarder le problème (1.15) comme une équation différentielle à valeurs dans H. En effet, U vérifie

$$\begin{cases} \mathrm{d}U_t = [AU_t + F(U_t)]\,\mathrm{d}t + \mathrm{d}W_t, & t \in [0,T], \\ U_0 = \phi. \end{cases} \tag{1.16}$$

La fonction F porte le nom d'opérateur de Nemytskii. Remarquons que la formulation précédente n'a de sens que si $U_t \in D(A)$, $\forall t \geq 0$. Cependant, soit $h \in D(A)$, alors

$$\langle -Ah, h\rangle_H = \int_0^\pi \left(\frac{\mathrm{d}h}{\mathrm{d}x}\right)^2 \mathrm{d}x \geq 0,$$

ce qui montre que A est dissipatif. De plus, $\forall h, f \in D(A)$, par une double intégration partie,

$$\langle Ah, f\rangle = \int_0^\pi \frac{\mathrm{d}^2 h}{\mathrm{d}x^2}(x) f(x)\mathrm{d}x = \int_0^\pi h(x)\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(x)\mathrm{d}x = \langle h, Af\rangle,$$

ce qui montre que A est autoadjoint. De plus, le domaine de $D(A)$ est dense dans H. En effet, définissons $\forall k \in \mathbb{N}^*$,

$$\psi_0(x) = 1/\sqrt{\pi}$$
$$\psi_k(x) = \sqrt{2/\pi}\cos(kx), \quad \forall k \in \mathbb{N}^*.$$

Alors $(\psi_k)_{k\in\mathbb{N}}$ forme un système orthonormal complet de $D(A)$ dense dans H. On en déduit que $(A, D(A))$ est m-dissipatif. Par le théorème de Hile-Yosida, nous en déduisons que $(A, D(A))$ est le générateur infinitésimal d'un semi-groupe de contraction $\{e^{tA}\}$ sur H (i.e d'un semi-groupe fortement continu et tel que $||e^{tA}||_{\mathscr{L}(H,H)} \leq 1$), voir par exemple [20] ou [70]. Il est alors possible de considérer une version affaiblie de (1.16), qui ne demande qu'aux solutions d'appartenir à H au lieu de $D(A)$:

$$U_t = e^{tA}x + \int_0^t e^{(t-s)A}F(U_s)\mathrm{d}s + \int_0^t e^{(t-s)A}\mathrm{d}W_s,$$

et l'on parle alors de mild solutions, ou solution allégée en français (et non pas douce ou bonne). Notons au passage que cette terminologie est également utilisée dans d'autres contextes, comme par exemple celui des EDP qui permet d'abaisser la régularité requise pour les fonctions solutions. Notons enfin que cette formulation affaiblie se justifie par une formule d'Itô qui permet de généraliser la formule de Duhamel valable dans un contexte déterministe.

EDSR en dimension infinie et solutions mild d'EDP

Dans ce manuscrit, nous allons rencontrer des EDSR en dimension infinie. En réalité, seul le processus de Wiener qui dirige l'EDS et la solution de cette EDS sont à valeurs dans des espaces de dimension infinie. Il est bien sûr possible de considérer des EDSR réellement en dimension infinie (voir par exemple [10]) mais toutes les EDSR (ou EDSRE) que nous rencontrerons au travers de ce document sont à valeurs réelles. Considérons par exemple l'EDSR à valeurs dans $\mathbb{R}$

$$Y_s = g(X_T) + \int_s^T f(r, X_r, Y_r, Z_r)\mathrm{d}r - \int_s^T Z_r \mathrm{d}W_r, \qquad s \in [t, T],$$

avec W processus de Wiener cylindrique à valeur dans Ξ. Y est à valeur dans $\mathbb{R}$ et $Z \in \mathscr{L}_2(\Xi, \mathbb{R})$. Le processus X prend ses valeurs dans un autre espace de Hilbert H et est solution mild de l'EDS

$$\begin{cases} \mathrm{d}X_s = [AX_s + F(s, X_s)]\,\mathrm{d}s + G(s, X_s), s \in [t, T], \\ X_t = x \in H, \end{cases} \tag{1.17}$$

où A est le générateur d'un semi-groupe fortement continu d'opérateurs linéaires bornés $\{e^{tA}\}$ dans H. Tout comme dans le cas des EDSR en dimension finie, ce problème stochastique permet d'étudier l'équation de Kolmogorov, $\forall t \in [0, T]$, $\forall x \in H$,

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} + \mathscr{L}u(t, x) + f(t, x, u(t, x), \nabla u(t, x)G(t, x)) = 0, \\ u(T, x) = g(x), \end{cases} \tag{1.18}$$

où ∇u est la dérivée de Gâteaux de u par rapport à x et $\mathscr{L}$ est l'opérateur différentiel du second ordre suivant

$$\mathscr{L}\phi(x) = \frac{1}{2}\,\mathrm{Tr}(G(t, x)G(t, x)^*\nabla^2\phi(x)) + \langle Ax + F(t, x), \nabla\phi(x)\rangle.$$

En effet, si $X^{t,x}$ désigne la solution de (1.17) et si $(Y^{t,x}, Z^{t,x})$ désigne la solution de (1.18) avec X remplacé par $X^{t,x}$ alors il est possible de montrer que $u(t, x) = Y_t^{t,x}$ est solution de (1.18). Dans le cadre de la dimension finie ou infinie, le terme "solution" peut revêtir plusieurs concepts. Il est possible de s'intéresser aux solutions classiques de (1.18) (voir par exemple [67]) mais cette approche demande d'avoir une régularité très forte sur les coefficients de l'EDS et de l'EDSR, c'est-à-dire au moins $\mathscr{C}^1$ en la variable temporelle et $\mathscr{C}^2$ en la variable d'espace. Une autre approche consiste à s'intéresser aux solutions de viscosité de (1.18). Cependant, si cette approche s'intègre bien au cadre de la dimension finie (voir par exemple [66]), elle est en revanche beaucoup plus difficile à appliquer en dimension infinie. De plus, dans la perspective d'appliquer les résultats aux problèmes de contrôle optimal, il est intéressant d'avoir au moins une dérivée de Gâteaux ∇u afin de pouvoir exprimer le contrôle optimal comme une fonction déterministe de $X^{t,x}$ (caractérisation du contrôle optimal par "feedback law"). Il est donc naturel d'introduire un nouveau concept de solutions, dites solutions mild, voir par exemple [14] et [38], ou dans le cadre des EDSR [33] et [34]. Ainsi, une solution de (1.18) est dite mild si elle vérifie, $\forall t \in [0, T]$, $\forall x \in H$,

$$u(t,x) = \mathscr{P}_{T-t}[g](x) + \int_t^T \mathscr{P}_{s-t}[f(s,\cdot,u(s,\cdot),\nabla u(s,\cdot)G(s,\cdot))](x)\mathrm{d}s,$$

où $\mathscr{P}_t$ est généré par $\mathscr{L}$, i.e. $\mathscr{P}_t[\phi](x) = \mathbb{E}\phi(X_t^x)$. Cette formulation se retrouve en appliquant la formule de la variation de la constante à (1.18). Remarquons que cette formulation a du sens dès que u admet une dérivée de Gâteaux par rapport à x. Les solutions mild se positionnent donc à mi-chemin entre les solutions classiques et les solutions de viscosité.

1.2 Etude des EDSRs ergodiques

Les EDSRs ergodiques ont été introduites par Fuhrman, Hu et Tessitore dans [31]. Une EDSR ergodique est une équation du type :

$$Y_t = Y_T + \int_t^T [f(Y_s, Z_s) - \lambda]\mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s, \quad 0 \le t \le T < +\infty. \tag{1.19}$$

La seule différence structurelle avec une EDSR en horizon infini réside dans l'apparition du terme λ dans la première intégrale. Ce λ fait partie des inconnues. Une solution de l'EDSR ergodique précédente est donc un triplet $(Y_t, Z_t, \lambda)_{t\ge 0}$. La méthode proposée par Fuhrman, Hu et Tessitore pour prouver l'existence d'une solution à (1.20) permet de traiter le cas où Y est à valeurs dans $\mathbb{R}$, W est un processus de Wiener cylindrique, f est un générateur markovien et est indépendant de y. Nous réécrivons l'EDSR ergodique qui en découle :

$$Y_t = Y_T + \int_t^T [f(X_s, Z_s) - \lambda]\mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s, \quad 0 \le t \le T < +\infty. \tag{1.20}$$

Lorsque f est une fonction strictement monotone en y, Briand et Hu ont montré dans [11] qu'il existe une unique solution à l'EDSR précédente. Les EDSR ergodiques présentent la particularité de ne plus être monotones en Y. La constante λ peut donc être vue comme un terme compensatoire qui permet d'avoir un contrôle sur la solution. En effet nous verrons plus tard qu'il existe une solution (Y^x, Z^x, λ) telle que $|Y_t^x| \le C(1 + |X_t^x|^p)$ pour un certain $p > 0$. Rappelons que si les EDSR en horizon fini permettent d'étudier les EDP paraboliques semi-linéaires et que les EDSR en horizon infini permettent d'étudier les EDP elliptiques semi-linéaires, les EDSR ergodiques, quant à elles, permettent d'étudier les EDP ergodiques semi-linéaires (au sens mild en dimension infinie ou au sens de la viscosité en dimension finie). Rappelons qu'une EDP ergodique est une EDP de la forme

$$F(x, v, \nabla v, \nabla^2 v) - \lambda = 0,$$

où λ fait partie des inconnues. Autrement dit, l'inconnue des EDP ergodiques est un couple (v, λ). Une EDP ergodique est homogène en temps et les solutions recherchées le sont donc aussi. Les EDSR ergodiques nous permettent de traiter le cas où F est semi-linéaire :

$$F(x, v, \nabla v, \nabla^2 v) = \mathscr{L}v + f(x, \nabla v),$$

avec, $\forall x \in H$

$$\mathscr{L}u(x) = \frac{1}{2}\operatorname{Tr}(GG^*\nabla^2 v(x)) + \langle Ax, x\rangle + f(x, \nabla v(x)).$$

Remarquons que F étant dans notre contexte indépendant de v, i.e. $F(x, v, \nabla v, \nabla^2 v) = F(x, \nabla v, \nabla^2 v)$, il ne peut y avoir unicité des solutions ergodiques car si (v, λ) est solution alors, $\forall c \in \mathbb{R}$, $(v + c, \lambda)$ est aussi solution. Cependant, sous de bonnes hypothèses il y a unicité de λ et unicité de v à addition d'une constante près. Les EDSR ergodiques sont une bonne alternative pour étudier les problèmes de contrôles stochastiques ergodiques, et permettent aussi d'étudier le comportement en temps long des solutions d'EDP paraboliques semi-linéaires, comme nous le verrons dans la suite de ce manuscrit.

1.2.1 Résultats connus

Revenons à l'article [31]. Les auteurs considèrent un générateur f Lipschitz en x et z et un processus $(X_t^x)_{t\geq 0}$ à valeurs dans un espace de Banach E solution de l'EDS suivante :

$$\begin{cases} \mathrm{d}X_t^x = [AX_t^x + F(X_t^x)]\mathrm{d}t + G\mathrm{d}W_t, \\ X_0 = x, \end{cases} \tag{1.21}$$

où, sans rentrer dans les détails, W est un processus de Wiener, A est le générateur d'un semi-groupe de contractions fortement continu noté $\left\{e^{tA}\right\}_{t\geq 0}$, F est mesurable et bornée, $A + F + \eta I$ est dissipatif et G est un opérateur linéaire borné. Ces hypothèses permettent d'obtenir la décroissance exponentielle des trajectoires pour les solutions de (1.21), c'est-à-dire :

$$|X_t^x - X_t^y| \leq e^{-\eta t}|x - y|, \quad t \geq 0, \quad \mathbb{P}\text{-p.s.}, \quad x, y \in E.$$

Puis, ils introduisent l'EDSR suivante en horizon infini, pour $\alpha > 0$:

$$Y_t^{x,\alpha} = Y_T^{x,\alpha} + \int_t^T [f(X_s^x, Z_s^{x,\alpha}) - \alpha Y_s^{x,\alpha}]\,\mathrm{d}s - \int_t^T Z_s^{x,\alpha}\mathrm{d}W_s, \tag{1.22}$$

où f est supposé Lipschitz. Il s'agit d'une EDSR en horizon infini dont le générateur est strictement monotone en y. Comme la structure des équations est markovienne, en définissant $v^\alpha(x) := Y_0^{x,\alpha}$, il découle par unicité des solutions de l'EDSR (1.22) que $v^\alpha(X_t^x) = Y_t^{x,\alpha}$. Le fait que $(Y_t^{x,\alpha})_{t\geq 0}$ soit un processus à valeurs réelles et que le générateur soit strictement monotone en y avec $\alpha > 0$ permet d'obtenir l'estimée suivante (voir l'article de Briand et Hu [11]) :

$$\alpha|v^\alpha(0)| \leq C. \tag{1.23}$$

En appliquant une formule d'Itô à $e^{-\alpha T}Y_T^{x,\alpha}$, il apparaît que le terme linéaire en αy ajouté au générateur dans l'EDSR ergodique permet d'annuler la condition terminale en faisant $T \to +\infty$. En utilisant la décroissance exponentielle des trajectoires et en utilisant le fait que f est Lipschitz en x et z, on peut montrer que la famille de fonctions $\{v^\alpha\}_{0<\alpha\leq 1}$ est équicontinue, $\forall 0 < \alpha \leq 1$, $\forall x, y \in E$,

$$|v^\alpha(x) - v^\alpha(y)| \leq C|x - y|. \tag{1.24}$$

Il est alors possible de réécrire (1.22) de la manière suivante :

$$Y_t^{x,\alpha} - Y_0^{0,\alpha} = Y_T^{x,\alpha} - Y_0^{0,\alpha} + \int_t^T [f(X_s^x, Z_s^{x,\alpha}) - \alpha(Y_s^{x,\alpha} - Y_0^{0,\alpha}) - \alpha Y_0^{0,\alpha}]\mathrm{d}s - \int_t^T Z_s^{x,\alpha}\mathrm{d}W_s, \tag{1.25}$$

afin de ne faire apparaître que des termes contrôlés grâce à (1.23) et (1.24). Ainsi, grâce à ces deux estimées (1.23) et (1.24), il est possible de construire, par une extraction diagonale une suite $\alpha_n \underset{n\to+\infty}{\longrightarrow} 0$ telle que

$$\begin{aligned}
&v^{\alpha_n}(x) - v^{\alpha_n}(0) \underset{n\to+\infty}{\longrightarrow} \overline{v}(x)\\
&\alpha_n v^{\alpha_n}(0) \underset{n\to+\infty}{\longrightarrow} \lambda.
\end{aligned}$$

Par des arguments usuels (théorème de convergence dominée, formule d'Itô), ils montrent alors que

$$(\overline{Y}_t^{x,\alpha_n} := \overline{v}^{\alpha_n}(X_t^x), Z_t^{x,\alpha_n}, \alpha_n Y_0^{0,\alpha_n} = v^{\alpha_n}(0)) \underset{n\to+\infty}{\longrightarrow} (\overline{Y}_t^x, Z_t^x, \lambda),$$

et que le membre de droite est bien solution de (1.20).

Nous voyons que la décroissance exponentielle des trajectoires permet de contrôler les accroissements de $(x \mapsto v^\alpha(x))$. Dans l'article [26], Debussche, Hu et Tessitore considèrent une diffusion X^x à valeurs dans un espace de Hilbert H solution de l'EDS suivante :

$$\begin{cases} \mathrm{d}X_t^x = [AX_t^x + F(X_t^x)]\mathrm{d}t + G\mathrm{d}W_t, \\ X_0 = x, \end{cases} \tag{1.26}$$

où cette fois, $A + \eta I$ est dissipatif, F est Lipschitz et borné. On dit alors que $A + F$ est faiblement dissipatif. A priori, $A + F$ n'est plus dissipatif et il est vain d'espérer obtenir la décroissance exponentielle des trajectoires. Cependant, en supposant que G soit inversible, il est possible d'établir, par des arguments de couplage, que pour toute fonction borélienne bornée ϕ,

$$|\mathbb{E}\phi(X_t^x) - \mathbb{E}\phi(X_t^y)| \leq C(1 + |x|^2 + |y|^2)e^{-\hat{\eta}t}. \tag{1.27}$$

Cette inégalité est établie par des arguments de couplage. Notons que cette inégalité implique que le processus est exponentiellement mélangeant (exponential mixing), i.e. il existe une unique mesure invariante μ telle que pour toute fonction $\phi : \mathbb{R}^d \to \mathbb{R}$ borélienne bornée,

$$\left|\mathbb{E}\phi(X_t^x) - \int_{\mathbb{R}^d} \phi(x)\mathrm{d}\mu(x)\right| \leq C(1 + |x|^2)e^{-\hat{\eta}t}.$$

De nombreux articles traitent de cette problématique, voir par exemple, [25], [51], [27], [63], [64], [12], [39], [28], [49], [50] ou [75]. La majoration (1.27) est dénommée "Basic coupling estimate". Il est même possible, lorsque les estimées obtenus ne dépendent pas de la constante de Lipschitz de F, d'étendre ce résultat par un argument de Girsanov au cas où F est seulement mesurable et borné et limite d'une suite de fonction Lispchitz uniformément bornée (i.e. $\exists (F_n)_{n\in\mathbb{N}}$, F_n Lipschitz $\forall n$, $\forall x \in H$, $F_n(x) \underset{n\to+\infty}{\longrightarrow} F(x)$, et $\sup_{n,x} |F_n(x)| < +\infty$). Dans ce cas, il n'y a plus existence et unicité forte des solutions de (1.26) mais il y a toujours existence et unicité en loi des solutions de martingale de l'EDS (1.26). Rappelons que dans le cadre des EDS en dimension infinie, le terme "solution de martingale" est l'équivalent des solutions faibles pour les EDS en dimension finie. L'expression solution faible pour les EDS en dimension infinie est réservée à un autre concept, voir [23]. Dans le contexte des solutions de martingale, le Basic coupling estimate s'énonce comme suit : si (X^x, W^x) et (X^y, W^y) sont deux solutions de martingale de (1.26) alors en notant E^x (respectivement E^y) l'espérance associée à la probabilité sous laquelle W^x (respectivement W^y) est un mouvement brownien, alors

$$|E^x\phi(X^x_t) - E^y\phi(X)| \leq C(1 + |x|^2 + |y|^2)e^{-\hat{\eta}t}. \tag{1.28}$$

La majoration (1.28) permet de contrôler uniformément les accroissements de $v^\alpha(x)$ mais ne permet pas a priori d'obtenir de l'équicontinuité pour la famille $v^\alpha(x)$. Pour pallier cet inconvénient, il faut en outre supposer que F est Gâteaux différentiable. Alors, comme G est inversible, il est possible d'appliquer une formule de Bismut Elworthy pour les EDSRs en dimension infinie démontrée dans [32] qui permet d'obtenir une estimée sur la dérivée de Gâteaux de v^α uniformément en α.

Nous avons déjà évoqué l'article [31] dans lequel ont été introduites les EDSR ergodiques. Richou dans [72] s'intéresse aux EDSR ergodiques en lien avec les EDP avec conditions de Neumann au bord. Cela revient à considérer l'EDSR ergodique, $\forall 0 \leq t \leq T < +\infty$,

$$Y_t = Y_T + \int_t^T [f(X^x_s, Z_s) - \lambda]\,\mathrm{d}s + \int_t^T [g(X^x_s) - \mu]\,\mathrm{d}K^x_s - \int_t^T Z_s \mathrm{d}W_s, \tag{1.29}$$

où (X^x, K^x) est la solution d'une EDS réfléchie dans un domaine borné convexe $G \subset \mathbb{R}^d$ à bord régulier, i.e. $\forall t \geq 0$,

$$\begin{cases} X^x_t = x + \int_0^t b(X^x_s)\mathrm{d}s + \int_0^t \nabla\phi(X^x_s)\mathrm{d}K^x_s + \int_0^t \sigma(X^x_s)\mathrm{d}W_s, \\ K^x_t = \int_0^t \mathbb{1}_{\{X^x_s \in \partial G\}}\mathrm{d}K^x_s. \end{cases}$$

Dans l'équation (1.29), si μ fait partie des données du problème, alors l'inconnue est le triplet $(Y_t, Z_t, \lambda)_{t\geq 0}$. On parle alors de problème ergodique. Inversement, si λ fait partie des données, alors l'inconnue est le triplet $(Y_t, Z_t, \mu)_{t\geq 0}$ et l'on parle de problème ergodique au bord (ou à la frontière). Le drift de l'EDS b est supposé dissipatif strict, (i.e. $\exists \eta > 0$, $\langle b(x) - b(y), x - y\rangle \leq -\eta|x-y|^2$). Cette condition suffit lorsque la matrice de diffusion de l'EDS est constante, cependant, lorsque cette condition n'est plus vérifiée, il est nécessaire d'imposer une condition plus forte reliant η, la constante de Lipschitz de σ et la constante de Lispchitz de f en z. Notons au passage que σ peut être dégénérée. Ces hypothèses sur l'EDS permettent d'obtenir la décroissance exponentielle des trajectoires. La stratégie de résolution des EDSR avec conditions de Neumann au bord est la suivante. D'abord, Richou considère le problème avec condition de Neumann au bord nulle, c'est-à-dire en imposant $g - \mu = 0$. On est donc ramené à étudier l'EDSRE, $\forall 0 \leq t \leq T < +\infty$,

$$Y_t = Y_T + \int_t^T [f(X^x_s, Z_s) - \lambda]\,\mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s. \tag{1.30}$$

Pour étudier cette EDSRE, l'auteur utilise la méthode développée par Fuhrman, Hu et Tessitore dans [31], qui consiste à approcher cette EDSRE par (1.22). On obtient donc une solution (Y, Z, λ) de l'EDSR ergodique (1.30). Reste à traiter le cas où les conditions de Neumann ne sont pas nulles. Pour cela, un théorème qui se trouve dans [52] nous donne l'existence d'un $\alpha \in \mathbb{R}$ tel que l'équation de Helmholtz avec condition de Neumann au bord suivante

$$\begin{cases} \Delta v(x) - \alpha v(x) = 0, \\ \frac{\partial v(x)}{\partial n} + g(x) - \mu = 0. \end{cases}$$

admette une solution $v \in \mathscr{C}^2(\overline{G})$, sous réserve que g soit assez régulière. En appliquant une formule d'Itô, on peut montrer que $(v(X_t^x), {}^t\nabla v(X_t^x)\sigma(X_t^x))$ est solution d'une EDSR avec condition de Neumann au bord. En utilisant l'invariance en y du générateur de l'EDSR ergodique que nous souhaitons résoudre, on peut alors trouver une solution (Y, Z, λ) de l'EDSR ergodique (1.29). Pour résoudre le problème ergodique à la frontière, l'auteur commence par montrer que $\mu \mapsto \lambda(\mu)$ est une fonction continue décroissante et telle que $\lambda \underset{\mu\to+\infty}{\longrightarrow} -\infty$, $\lambda \underset{\mu\to-\infty}{\longrightarrow} +\infty$. Le théorème des valeurs intermédiaires permet alors de conclure. Notons que l'article de Barles et Da Lio [4] traite de ce problème dans un contexte déterministe et pour des EDP possiblement non linéaires.

D'autres articles sur les EDSR ergodiques ont vu le jour par la suite. Nous n'en donnons pas une présentation détaillée ici. L'article [17] traite du cas des EDSR ergodiques dont le bruit est donné par une chaîne de Markov uniformément ergodique à espace d'état dénombrable. Dans [16], Cohen et Fedyashov traite du cas des EDSR ergodiques à sauts et dont les coefficients dépendent du temps. Enfin, dans [18], Hu M. et Wang s'intéressent au cas des EDSR ergodiques dirigées par un G-mouvement brownien.

1.2.2 Résultats nouveaux

Le chapitre 2 de ce manuscrit présente les résultats obtenus concernant des résultats d'existence et d'unicité sur les solutions de l'EDSR ergodique (1.29) lorsque la diffusion X^x est solution de l'EDS réfléchie dans $\overline{G}$ suivante

$$\begin{cases} X_t^x = x + \int_0^t (d+b)(X_s^x)\mathrm{d}s + \int_0^t \nabla\phi(X_s^x)\mathrm{d}K_s^x + \int_0^t \sigma(X_s^x)\mathrm{d}W_s, \\ K_t^x = \int_0^t \mathbb{1}_{\{X_s^x\in\partial G\}}\mathrm{d}K_s^x. \end{cases} \tag{1.31}$$

Contrairement à [72], le drift $d+b$ est faiblement dissipatif et $\overline{G}$ peut être non borné. La matrice de diffusion est inversible, bornée et dont l'inverse est bornée. Sous ces hypothèses il y a bon espoir d'obtenir le basic coupling estimate (1.28). Cependant, ce résultat étant délicat à établir pour la diffusion réfléchie X^x, nous considérons le problème pénalisé associé à (1.31),

$$X_t^{x,n} = x + \int_0^t (d+b+F_n)(X_s^{x,n})\mathrm{d}r + \int_0^t \sigma(X_s^{x,n})\mathrm{d}W_s,$$

où $F_n(x) = -2n(x - \Pi(x))$ avec Π l'opérateur de projection sur le convexe $\overline{G}$. F_n est le terme de pénalisation, qui agit comme une force de rappel vers le convexe $\overline{G}$. Le terme F_n étant dissipatif et d étant dissipatif strict, le terme $d + F_n$ est dissipatif strict, indépendamment de n. Comme le Basic coupling estimate ne dépend du terme dissipatif qu'au travers de sa constante de dissipativité, nous pouvons utiliser le Basic coupling estimates au processus pénalisé et obtenir des estimées indépendantes de n. Lorsque $n \to +\infty$, la limite de $X^{x,n}$ nous donne X^x. Il apparaît alors qu'une condition supplémentaire est nécessaire sur σ, $\exists\Lambda \geq 0$, $\forall x, y \in \mathbb{R}^d$,

$$|\left(y, \sigma(x+y) - \sigma(x)\right)| \leq \Lambda|y|,$$

et $\exists\lambda > 0$,

$$2(\lambda - \lambda^2\Lambda^2) > |||\sigma^{-1}|||^2.$$

En dimension 1, cette condition dit que les fluctuations de σ ne doivent pas être trop grandes et que σ doit être très non dégénérée, i.e. $|||\sigma^{-1}|||$ petit. Remarquons que dès que σ est constante, cette condition est immédiatement vérifiée. Cela n'est cependant pas suffisant, car afin de pouvoir appliquer le Basic coupling estimate dans notre contexte, il nous faut plus de régularité sur les coefficients de l'EDS afin que le terme perturbateur soit bien limite simple de fonctions Lipschitz uniformément bornées. Pour contourner ce problème, il est possible de régulariser les coefficients de l'EDS. En notant ε l'indice de régularisation des coefficients, l'EDS considérée est alors

$$X_t^{x,n,\varepsilon} = x + \int_0^t \left(d_\varepsilon + b_\varepsilon + (F_n)_\varepsilon\right)(X_s^{x,n,\varepsilon})\mathrm{d}s + \int_0^t \sigma(X_s^{x,n,\varepsilon})\mathrm{d}W_s,$$

et l'EDSR monotone servant à approcher l'EDSR ergodique devient :

$$Y_s^{x,\alpha,n,\varepsilon} = Y_T^{x,\alpha,n,\varepsilon} + \int_s^T \left[f(X_s^{x,\alpha,n,\varepsilon}) - \alpha Y_s^{x,\alpha,n,\varepsilon}\right]\mathrm{d}s - \int_s^T Z_s^{x,\alpha,n,\varepsilon}\mathrm{d}W_s.$$

Pour pouvoir appliquer un résultat de Ma et Zhang dans [54], il faut que d soit Lipschitz, ce qui n'est a priori pas le cas ici. En approchant d par une "bonne" suite de fonction Lipschitz d_m, il est possible de contourner ce problème et d'appliquer la formule de représentation de Ma et Zhang dans [54] afin d'obtenir un contrôle sur les accroissements de $v^{\alpha,n,\varepsilon}$ et l'équicontinuité de $v^{\alpha,n,\varepsilon}$. Les bornes obtenues sont indépendantes des paramètres α, n et ε. Par une procédure d'extraction diagonale, il est donc possible de prouver l'existence d'une solution à l'EDSR ergodique (1.30).

Passons au cas des conditions de Neumann non nulles. La technique utilisée dans [72] pour résoudre l'EDSR ergodique (1.29) ne semble pas fonctionner car il ne semble pas exister dans la littérature de résultat concernant l'existence d'une solution régulière au problème de Helmholtz avec condition de Neumann dans le cas où $\overline{G}$ est non borné. Notons au passage qu'il n'est pas indispensable d'avoir une solution au problème de Helmoltz avec condition de Neumann. En fait seules les conditions au bord sont importantes et n'importe quel autre problème déterministe convient à partir du moment où il nous fournit une fonction solution $\mathscr{C}^2(\overline{G})$. Une fois le résultat d'existence obtenu lorsque les conditions de Neumann sont nulles, nous pouvons l'appliquer à un problème de contrôle ergodique dont le coût est

$$I(x,u) = \limsup_{T\to+\infty} \frac{1}{T}\mathbb{E}_T^u\left[\int_0^T L(X_s^x, u_s)\mathrm{d}s\right],$$

u étant le contrôle, i.e. un processus adapté à valeur dans un espace métrique séparable. $\mathbb{E}_T^u$ est l'espérance relative à la probabilité $\mathbb{P}_T^u$ sous laquelle $\left(W_t + \int_0^t R(u_s)\mathrm{d}s\right)_{t\in[0,T]}$ est un mouvement brownien. En définissant l'hamiltonien suivant

$$f_0(x,z) = \inf_{u_0\in U}\left\{L(x,u_0) + z\sigma^{-1}R(u_0)\right\}, \tag{1.32}$$

et en étudiant l'EDSR ergodique dont le générateur est f_0, il est possible de montrer que pour tout contrôle admissible u,

$$I(x,u) \geq \lambda,$$

et que si l'infimum est atteint dans 1.32, alors il existe un contrôle optimal $\overline{u}$ tel que $I(x,\overline{u}) = \lambda$ et de plus le contrôle optimal est fonction déterministe de la diffusion réfléchie X^x.

Ce chapitre est organisé comme suit. Dans la partie 2.2, nous établissons des résultats concernant les EDS en environnement faiblement dissipatif. Le résultat important de cette partie est le Basic coupling estimate car la partie du drift dissipative n'étant plus linéaire contrairement au cas de la dimension infinie et la matrice de diffusion n'étant plus constante, il est nécessaire de revoir le couplage permettant d'obtenir un tel résultat. Dans la partie 2.3, nous établissons un résultat d'existence et d'unicité pour les EDSRE sans condition de Neumann. Dans la partie 2.4, nous traitons de l'existence et de l'unicité des EDSRE avec condition de Neumann.

1.3 Comportement en temps long des solutions d'EDP paraboliques semi-linéaires

Cette partie se décline en trois sous-parties. Tout d'abord nous présentons la problématique générale et les résultats connus. Puis nous présentons les résultats obtenus concernant le comportement en temps long des solutions mild d'EDP semi-linéaires en dimension infinie. Enfin, en dimension finie et en imposant des conditions de Neumann à l'EDP, nous établissons des résultats similaires pour les solutions de viscosité d'une EDP parabolique semi-linéaire en adaptant les arguments utilisés en dimension infinie.

1.3.1 Résultats connus

Le comportement en temps long des solutions d'EDP paraboliques a été largement étudié par des méthodes analytiques, et cela quel que soit le cadre d'étude : équation du premier ou second ordre, pas de condition, condition de Dirichlet ou de Neumann au bord.

Supposons que u est solution de l'EDP

$$\begin{cases} \frac{\partial u}{\partial t} + F(x, u, \nabla u, \nabla^2 u) = 0, \\ u(0, x) = u_0(x), \end{cases} \tag{1.33}$$

et considérons (v, λ) solution de l'EDP ergodique associée

$$F(x, v, \nabla v, \nabla^2 v) - \lambda = 0. \tag{1.34}$$

Alors le résultat typique du comportement en temps long pour ce genre d'équation dit qu'il existe un (v, λ) solution de (1.34) et $L \in \mathbb{R}$,

$$\frac{u(T, x)}{T} \underset{T \to +\infty}{\longrightarrow} \lambda \tag{1.35}$$

$$u(T, x) - \lambda T - v(x) \underset{T \to +\infty}{\longrightarrow} L, \tag{1.36}$$

uniformément sur les ensembles bornés. Remarquons tout de suite que (1.36) $\Rightarrow$ (1.35). La constante λ ne dépend pas de u_0 alors que L dépend de u_0. Le troisième comportement que l'on peut espérer obtenir est d'avoir une vitesse pour la limite de (1.36) :

$$|u(T, x) - \lambda T - v(x) - L| \leq C_x e^{-\eta T}, \tag{1.37}$$

où C_x est une constante qui dépend de x. Notons au passage que (1.37) $\Rightarrow$ (1.36) $\Rightarrow$ (1.35). Les deux premiers comportements apparaissent fréquemment dans la littérature, le troisième apparaît beaucoup plus rarement. La quantité $\lambda T - v(x) - L$ porte parfois le nom de front mouvant ("propagating front" en anglais).

Dans [61], Namah et Roquejoffre étudient le comportement en temps long des solutions classiques $u : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ de

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + f(x, \nabla u), \\ u(0, x) = u_0(x). \end{cases}$$

En supposant que les coefficients sont périodiques, C^2 avec des bornes sur les dérivées partielles et en supposant qu'il existe $0 < m < M$ tel que $m < f(x, z) < M$, les auteurs réussissent à obtenir une vitesse de convergence exponentielle.

Dans [35], Fujita, Ishii et Loreti, étudient le comportement en temps long des solutions de viscosité $u : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ de l'EDP semi-linéaire parabolique

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u - \langle \alpha x, \nabla u \rangle - H(\nabla u) + f(x), \\ u(0, x) = u_0(x), \end{cases}$$

avec $\alpha > 0$. Notons la présence du terme dissipatif $\langle \alpha x, \nabla u \rangle$ nécessaire lorsqu'on étudie le comportement en temps long dans des domaines non bornés. Notons la structure particulière des termes non linéaires qui est découplée : $f(x) - H(\nabla u)$. En supposant que f est Hölder à croissance dominée par $e^{\mu|x|^2}$ pour $\mu > 0$, que u_0 est continue et à croissance dominée par $e^{\mu|x|^2}$ et en supposant que H est Lipschitz, les auteurs réussissent par des méthodes analytiques, à établir la convergence au sens des deux premiers comportements. Le cas de H localement Lipschitz est également traité mais il faut alors supposer que f et u_0 sont Lipschitz. Sans cette hypothèse de dissipativité, Ishiii dans [44] établit le premier et le deuxième comportement pour la solution de viscosité de l'EDP du premier ordre

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = -H(x, \nabla u(t,x)) \\ u(0,x) = u_0(x) \end{cases}$$

sous des hypothèses de coercivité pour $H(\cdot, p)$ et de convexité pour $H(x, \cdot)$. Notons que ce même problème avec des hypothèses supplémentaires sur la périodicité des coefficients avait déjà été traité dans [6] par Barles et Souganidis.

Notons qu'il existe des résultats de convergence pour les solutions de viscosité des EDP du premier et second ordre avec croissance quadratique en le gradient de la solution. L'article [62] est l'un des premiers papiers traitant du comportement en temps long des solutions de viscosité d'une EDP du premier ordre. Fujita et Loreti dans [36] étudient le cas où la dépendance en le gradient est explicite et quadratique et avec un terme dissipatif dans l'équation. Mentionnons également [43] pour le cas d'une EDP du second ordre avec croissance quadratique en le gradient et [73] dans lequel le cas du comportement en temps long des solutions classiques d'une EDP semi-linéaire est traité par des méthodes probabilistes. Notons que dans ces papiers, seuls des résultats du premier type et deuxième type sont établis, les auteurs n'obtiennent pas de vitesse de convergence. Mentionnons également l'article [18] qui établit des résultats de convergences pour des équations de Bellman complètement non-linéaires.

1.3.2 Résultat nouveaux : Étude du comportement en temps long des solutions mild d'EDP paraboliques semi-linéaires en dimension infinie

Dans le chapitre 4, nous présentons les résultats obtenus concernant le comportement en temps long des solutions mild des EDP parabolique semi-linéaires du type :

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \mathscr{L}u(t,x) + f(x, {}^t\nabla u(t,x)G), & \forall t \in \mathbb{R}_+, \forall x \in H, \\ u(0,x) = g(x), \quad \forall x \in H, \end{cases} \tag{1.38}$$

où

$$\mathscr{L}h(t,x) := \frac{1}{2}\operatorname{Tr}(GG^*\nabla^2 h(t,x)) + \langle Ax + F(x), \nabla h(t,x) \rangle,$$

et où H est un espace de Hilbert. La démarche probabiliste est la suivante. Soit $(Y^{T,t,x}, Z^{T,t,x})$ la solution de l'EDSR, $\forall 0 \leq t \leq s \leq T$,

$$Y_s^{T,t,x} = g(X_T^{t,x}) + \int_s^T f(X_r^{t,x}, Z_r^{T,t,x})\mathrm{d}r - \int_s^T Z_r^{T,t,x}\mathrm{d}W_r, \tag{1.39}$$

où $X^{t,x}$ est solution mild de

$$\mathrm{d}X_s^{t,x} = \left[AX_s^{t,x} + F(X_s^{t,x})\right]\mathrm{d}s + G\mathrm{d}W_s.$$

Alors en définissant

$$u_T(t,x) = Y_t^{T,t,x}, \tag{1.40}$$

il est connu que $u_T(t,x)$ est solution mild de

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) + \mathscr{L}u(t,x) + f(x, {}^t\nabla u(t,x)G) = 0, & \forall t \in [0,T], \forall x \in H, \\ u(T,x) = g(x), \quad \forall x \in H. \end{cases} \tag{1.41}$$

Effectuons le changement de temps suivant $\widetilde{u}(t,x) = u_T(T-t,x)$, alors $\widetilde{u}$ est solution mild de

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \mathscr{L}u(t,x) + f(x, {}^t\nabla u(t,x)G), & \forall t \in [0,T], \forall x \in H, \\ u(0,x) = g(x) \quad \forall x \in H. \end{cases} \tag{1.42}$$

Si de plus, nous avons l'unicité des solutions mild pour l'EDP précédente, alors par unicité, $\widetilde{u}$ est solution de (1.38). Or $\forall T \geq 0$, $\widetilde{u}(T,x) = u_T(0,x) = Y_0^{T,x}$. Donc pour étudier le comportement en temps long de la solution de (1.38), il suffit d'étudier le comportement en temps long de la solution $Y^{T,x}$ de l'EDSR (1.39) prise au temps initial 0. Considérons l'EDP ergodique associée à (1.38) :

$$\mathscr{L}v(x) + f(x, {}^t\nabla v(x)G) - \lambda, \quad \forall x \in H. \tag{1.43}$$

Supposons que cette EDP admette une unique solution mild $(v(x), \lambda)$ (à constante additive près pour v). Alors nous obtenons les trois comportements décrits plus haut, avec vitesse exponentielle pour la convergence, plus précisément $\exists L \in \mathbb{R}$,

$$\begin{aligned} &\left| \frac{u_T(0,x)}{T} - \lambda \right| \leq \frac{C(1+|x|^\mu)}{T}, \\ &u_T(0,x) - \lambda T - v(x) \underset{T\to+\infty}{\longrightarrow} L, \\ &|u_T(0,x) - \lambda T - v(x) - L| \leq C(1+|x|^\mu)e^{-\hat{\eta}T}. \end{aligned}$$

Discutons à présent des hypothèses sous lesquelles nous obtenons ces résultats. L'opérateur A est dissipatif et génère un semi-groupe stable de contraction e^{tA} qui est de plus Hilbert-Schmidt, G est un opérateur inversible linéaire borné, que F est Lipschitz borné et que $f(\cdot, z)$ est continu avec croissance polynomiale de degré μ et $f(x, \cdot)$ Lipschitz (uniformément en x). Remarquons tout de suite que F ne joue en réalité aucun rôle, au moins d'un point de vue déterministe puisque

$$\langle F(x), \nabla u(x) \rangle + f(x, \nabla u(x)G) = \widetilde{f}(x, \nabla u(x)G),$$

avec $\widetilde{f}(x,p) = \langle F(x), pG^{-1} \rangle + f(x,p)$. Il est facile de vérifier que $\widetilde{f}$ vérifie les mêmes hypothèses que f et on peut donc toujours se ramener au cas où $F \equiv 0$ quitte à remplacer f par $\widetilde{f}$. L'intérêt de l'approche probabiliste est qu'elle permet d'utiliser le Basic coupling estimate, qui s'avère être un outil puissant pour étudier le comportement dans notre contexte. Remarquons que l'hypothèse de non-dégénérescence sur G est cruciale pour établir le Basic coupling estimate et il semble délicat d'établir un résultat analogue lorsque G est dégénéré. Discutons brièvement des méthodes utilisées pour établir ces résultats. Tout d'abord, considérons l'EDSR ergodique suivante, $\forall 0 \leq s \leq T < +\infty$,

$$Y_s^x = Y_T^x + \int_s^T [f(X_s^x, Z_s^x) - \lambda]\, \mathrm{d}s - \int_s^T Z_s^x \mathrm{d}W_s. \tag{1.44}$$

Alors

$$(v(x) = Y_0^x, \lambda) \tag{1.45}$$

est la solution mild de (1.43). Rappelons que par unicité des solutions pour les EDSR $Y_s^x = v(X_s^x)$, $\forall s$, $\mathbb{P}$-p.s.. Effectuons la différence entre l'EDSR (1.39) et (1.44), alors $\forall T \geq 0$,

$$Y_0^{T,x} - Y_0^x - \lambda T = g(X_T^x) - v(X_T^x) + \int_0^T \left(f(X_s^x, Z_s^{T,x}) - f(X_s^x, Z_s^{T,x})\right) \mathrm{d}s \\ - \int_0^T \left(Z_s^{T,x} - Z_s^x\right) \mathrm{d}W_s.$$

En utilisant un argument de Girsanov pour faire disparaître les termes en Z de l'équation précédente il ne reste plus que des quantités facilement contrôlables. On obtient alors que

$$|Y_0^{T,x} - Y_0^x - \lambda T| \leq C(1 + |x|^\mu) \tag{1.46}$$

Remarquons que l'hypothèse sur le caractère Lipschitzien de f en z est cruciale afin de pouvoir effectuer cette transformation. Il suffit alors d'utiliser la décomposition

$$\left|\frac{Y_0^{T,x}}{T} - \lambda\right| \leq \left|\frac{Y_0^{T,x} - Y_0^x - \lambda}{T}\right| + \left|\frac{Y_0^x}{T}\right|,$$

et d'utiliser les identifications (1.40) et (1.45) pour conclure. Le deuxième et le troisième comportement sont un peu plus délicats à établir. Tout d'abord définissons

$$w_T(0,x) = u_T(0,x) - \lambda T - v(x).$$

Remarquons que $\{w(T,\cdot)\}_T$ est une famille de fonctions continues et bornées uniformément en temps grâce à (1.46). Il est donc possible d'extraire une suite $T_i \nearrow +\infty$ telle que pour tout $x \in D$, D étant un sous-ensemble dénombrable dense de H,

$$w_{T_i}(0,x) \to w(x),$$

pour une certaine fonction w définie sur D. Cette convergence n'est pas entièrement satisfaisante car elle est relative à une suite $(T_i)_i$ et n'est pas valable pour tout $x \in H$. Pour pallier cet inconvénient, nous évaluons le gradient $\nabla w_T(0,x)$ et nous montrons qu'il peut être borné uniformément en temps T. La fonction w peut donc être étendue à tout l'espace en une fonction continue w et telle que $\forall x \in H$, $\lim_i w_{T_i}(0,x) = w(x)$. De plus en utilisant le Basic coupling estimate, il est facile de voir que pour tout $x, y \in H$,

$$|w_T(0,x) - w_T(0,y)| \leq C(1 + |x|^2 + |y|^2)e^{-\hat{\eta}T}.$$

Cette dernière estimée nous permet de montrer que $w = L$ est constante. De plus en utilisant ce dernier résultat, le Basic coupling estimate grâce à des transformations de Girsanov et la structure des EDSR qui nous permet d'exprimer $w_{T+S}(0,x)$ en fonction de $w_T(0,x)$ via une transformation par un semi-groupe, on peut finalement montrer que la convergence a lieu pour $T \to +\infty$ et pas seulement pour une suite $(T_i)_i$., i.e, $\forall x \in H$,

$$w_T(0,x) \underset{T \to +\infty}{\longrightarrow} L.$$

Finalement, avec les mêmes arguments, il est possible d'obtenir une vitesse exponentielle pour cette convergence.

Ce chapitre est organisé de la façon suivante. Tout d'abord, dans la partie 4.3 nous effectuons quelques rappels concernant les EDS, EDSR et EDSRE utilisées ainsi que leurs liens avec les solutions mild de problèmes paraboliques ou elliptiques associés. Dans la partie 4.4, nous démontrons les résultats obtenus concernant le comportement en temps long des solutions mild. Enfin dans la dernière partie 4.5, nous appliquons les résultats obtenus à un problème de contrôle optimal, ce qui nous permet d'obtenir une asymptote pour le coût d'un problème de contrôle stochastique en horizon fini.

1.3.3 Résultats nouveaux : Étude du comportement en temps long des solutions de viscosité d'EDP paraboliques semi-linéaires avec conditions de Neumann au bord

Dans le chapitre 5, nous présentons les résultats obtenus concernant le comportement en temps long des solutions de viscosité d'EDP paraboliques semi-linéaires avec conditions de Neumann au bord du type

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \mathscr{L}u(t,x) + f(x, {}^t\nabla u(t,x)G), \quad \forall t \in \mathbb{R}_+, \forall x \in \overline{G}, \\ \frac{\partial u(t,x)}{\partial n} + g(x) = 0, \quad \forall t \in \mathbb{R}_+, \forall x \in \partial\overline{G}, \\ u(0,x) = h(x), \quad \forall x \in \overline{G}, \end{cases} \tag{1.47}$$

où

$$\mathscr{L}h(t,x) := \frac{1}{2}\operatorname{Tr}(\sigma{}^t\sigma\nabla^2 h(t,x)) + \langle b(x), \nabla h(t,x)\rangle,$$

où $\overline{G} = \{\phi > 0\}$ est un convexe borné à bord régulier ayant pour normale intérieure $\nabla\phi(x)$. Supposons que b soit Lipschitz, que σ soit inversible et que $f(\cdot, z)$ soit continu avec croissance polynomiale de degré μ et $f(x, \cdot)$ Lipschitz (uniformément en x). Dans ce travail, nous adoptons les arguments présentés dans la partie précédente à ce nouveau cadre de travail. Un des arguments essentiels dans le travail précédent était l'application du Basic coupling estimate. Cependant si nous considérons le problème stochastique associé à l'EDP parabolique précédente, nous obtenons

$$X_s^x = x + \int_0^s b(X_r^x)\mathrm{d}r + \int_0^s \nabla\phi(X_s^x)\mathrm{d}K_s^x + \int_0^s \sigma(X_r^x)\mathrm{d}W_r,$$
$$K_s^x = \int_0^s \mathbb{1}_{\{X_r^x \in \partial\overline{G}\}}\mathrm{d}K_r^x,$$

pour la partie EDS et

$$Y_s^{T,x} = h(X_T^x) + \int_s^T f(X_s^x, Z_s^{T,x})\mathrm{d}s + \int_s^T g(X_s^x)\mathrm{d}K_s^x - \int_s^T Z_s^{T,x}\mathrm{d}W_s,$$

alors il n'est pas possible d'obtenir de Basic coupling estimate car le drift b n'est pas faiblement dissipatif. Cependant, si nous définissons

$$\widetilde{b}(x) = -x + [b(\Pi(x)) + \Pi(x)],$$

où Π désigne la projection sur $\overline{G}$, alors $\widetilde{b}$ coïncide avec b sur $\overline{G}$ et $\widetilde{b} : \mathbb{R}^d \to \mathbb{R}^d$ est faiblement dissipative. On peut même faire rentrer la partie non dissipative du drift dans la partie non linéaire de l'EDP en utilisant la même astuce que dans le travail précédent. Finalement, on peut toujours se ramener au cas où $b = -x$ quitte à remplacer f par $\widetilde{f}(x,z) = f(x,z) + \langle b(\Pi(x)) + \Pi(x), z\sigma^{-1}\rangle$. Cependant cela n'est pas encore suffisant pour pouvoir obtenir le Basic coupling estimate. Comme il est difficile d'établir un Basic coupling estimate pour une diffusion réfléchie, nous pénalisons le problème par $F_n(x) = -2n(x - \Pi(x))$ et nous régularisons cette fonction pour pouvoir appliquer le Basic coupling estimates. En utilisant des résultats de stabilité pour les EDSR et en prenant soin d'effectuer des passages à la limite licites, nous obtenons les mêmes résultats que dans la section précédente, avec notamment une vitesse exponentielle pour la convergence de la solution vers son asymptote, i.e., $\forall x \in \overline{G}$,

$$|u(T,x) - \lambda T - v(x) - L| \leq Ce^{-\hat{\eta}T}.$$

Ce chapitre est organisé de la façon suivante. Tout d'abord, dans la partie 5.3 nous effectuons quelques rappels concernant les SDE, EDSR et EDSRE utilisées ainsi que leurs liens avec les solutions mild de problèmes paraboliques ou elliptiques associés. Dans la partie 5.4, nous démontrons les résultats obtenus concernant le comportement en temps long des solutions de viscosité. Enfin dans la dernière partie 5.5, nous appliquons les résultats obtenus à un problème de contrôle optimal, ce qui nous permet d'obtenir une asymptote pour le coût d'un problème de contrôle stochastique en horizon fini.

Première partie

Etude des EDSRs ergodiques

Chapitre 2

EDSRs ergodiques et EDPs avec conditions de Neumann au bord en environnement faiblement dissipatif

Résumé: Nous étudions une classe d'EDSR ergodiques reliées aux EDP avec conditions de Neumann au bord. L'aléa du générateur est donné par un processus solution d'une EDS dont le drift est faiblement dissipatif et dont la matrice de diffusion est inversible et bornée. De plus, ce processus est réfléchi dans un sous-ensemble convexe de $\mathbb{R}^d$ qui n'est pas nécessairement borné. Nous étudions le lien existant entre cette classe d'EDSRE et les EDP et nous appliquons nos résultats à un problème de contrôle optimal ergodique.

Mots clés: Équation différentielle stochastique rétrograde; drift faiblement dissipatif; conditions de Neumann; EDP ergodique; problème de contrôle optimal ergodique.

Abstract: We study a class of ergodic BSDEs related to PDEs with Neumann boundary conditions. The randomness of the driver is given by a forward process under weakly dissipative assumptions with an invertible and bounded diffusion matrix. Furthermore, this forward process is reflected in a convex subset of $\mathbb{R}^d$ not necessarily bounded. We study the link of such EBSDEs with PDEs and we apply our results to an ergodic optimal control problem.

Key words: Backward stochastic differential equations; weakly dissipative drift; Neumann boundary conditions; ergodic partial differential equations; optimal ergodic control problem.

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under the title: Ergodic BSDEs and related PDEs with Neumann boundary conditions under weak dissipative assumptions.

2.1 Introduction

In this paper we study the following ergodic backward stochastic differential equation (EBSDE in what follows) in finite dimension and in infinite horizon: $\forall t, T \in \mathbb{R}_+, \quad 0 \leq t \leq T < +\infty$:

$$Y_t^x = Y_T^x + \int_t^T [\psi(X_s^x, Z_s^x) - \lambda]\mathrm{d}s + \int_t^T [g(X_s^x) - \mu]\mathrm{d}K_s^x - \int_t^T Z_s^x \mathrm{d}W_s, \tag{2.1}$$

where the given data satisfy:
- W is an $\mathbb{R}^d$-valued standard Brownian motion;
- $G = \{\phi > 0\}$ is an open convex subset of $\mathbb{R}^d$ with smooth boundary;
- $x \in G$;
- X^x is a $\overline{G}$-valued process starting from x, and K^x is a non decreasing real valued process starting from 0 such that the pair (X^x, K^x) is a solution of the following reflected stochastic differential equation (SDE in what follows):

$$X_t^x = x + \int_0^t f(X_s^x)\mathrm{d}s + \int_0^t \sigma(X_s^x)\mathrm{d}W_s + \int_0^t \nabla\phi(X_s^x)\mathrm{d}K_s^x, \quad t \geq 0,$$
$$K_t^x = \int_0^t \mathbb{1}_{\{X_s^x \in \partial G\}}\mathrm{d}K_s^x, \quad K_\cdot^x \ \text{ is non-decreasing,}$$

- $\psi : \mathbb{R}^d \times \mathbb{R}^{1\times d} \to \mathbb{R}$ is measurable and $g : \mathbb{R}^d \to \mathbb{R}$ is measurable;
- λ and μ belong both to $\mathbb{R}$. If λ is given then μ is unknown and if μ is given then λ is unknown.

Therefore, the unknown is either the triplet (Y^x, Z^x, λ) if μ is given or the triplet (Y^x, Z^x, μ) if λ is given, where:
- Y^x is a real-valued progressively measurable process;
- Z^x is an $\mathbb{R}^{1\times d}$-valued progressively measurable process.

We recall that a function $h : \mathbb{R}^d \to \mathbb{R}^d$ is said to be strictly dissipative if there exists a constant $\eta > 0$ such that, $\forall x, y \in \mathbb{R}^d$,

$$(h(x) - h(y), x - y) \leq -\eta|x - y|^2.$$

Richou in the paper [72] studied the case when $\overline{G}$ is bounded and with the assumptions that f and σ are Lipschitz and:

$$\sup_{x,y\in\overline{G}, x\neq y} \left\{ \frac{{}^t(x-y)(f(x)-f(y))}{|x-y|^2} + \frac{\mathrm{Tr}[(\sigma(x)-\sigma(y))\,{}^t(\sigma(x)-\sigma(y))]}{2|x-y|^2} \right\} < -K_{\psi,z}K_\sigma$$

where $K_{\psi,z}$ is the Lipschitz constant of ψ in z and K_σ is the Lipschitz constant of σ. Note that this assumption implies that f is strictly dissipative. However this hypothesis on f is not very natural because it supposes a dependence between parameters of the problem. Thanks to this condition it is possible to establish one of the key results: the strong estimate on the exponential decay in time of two solutions of the forward equation starting from different points. Indeed, it is used to construct, by a diagonal procedure, a solution to the EBSDE. Note that, in this work, $\overline{G}$ is assumed to be bounded.

In the paper [26], Debussche, Hu and Tessitore were concerned with the study of EBSDE in a weakly dissipative environment. This means that the driver of the forward process is assumed to be the sum of a strictly dissipative term and a perturbation term which is Lipschitz and bounded. In their infinite dimensional framework, they supposed that the dissipative term is linear. In addition, σ is constant, and the forward process is not reflected. Finally the coefficients of the forward process are assumed to be Gâteaux differentiable to obtain an estimate which is needed to prove the existence of a solution in this framework. In this context, the weaker assumption on f makes the strong estimate on the exponential decay in time of two solutions of the forward equation impossible. However it is possible to substitute this result by a weaker result, called "basic coupling estimate" which involves the Kolmogorov semigroups of the forward process X^x and which is enough to prove the existence of a solution to the EBSDE.

In this paper we extend the framework of [72] to the case of an unbounded domain $\overline{G}$ for a driver weakly dissipative. Namely, we assume that $f = d + b$ where d is locally Lipschitz and dissipative with polynomial growth and b is Lipschitz and bounded. The price to pay is that σ is assumed to be Lipschitz, invertible and such that σ and σ^{-1} are bounded. We do not need more regularity than continuous coefficients for this study, because we treat this problem by a regularization procedure. As the basic coupling estimate of [26] holds for a non reflected process, we start by studying the following forward process, $\forall t \geq 0$,

$$V_t^x = x + \int_0^t f(V_s^x)\mathrm{d}s + \int_0^t \sigma(V_s^x)\mathrm{d}W_s,$$

with f and σ defined as before. We show that the coupling estimate still holds in our framework with constants which depend on d only through its dissipativity coefficient. Once this is established, we apply this result to establish existence and uniqueness (of λ) of solutions to the following EBSDE:

$$Y_t^x = Y_T^x + \int_t^T [\psi(V_s^x, Z_s^x) - \lambda]\mathrm{d}s - \int_t^T Z_s^x \mathrm{d}W_s, \ \forall 0 \leq t \leq T < +\infty. \tag{2.2}$$

Then we want to obtain the same result when the process $V_\cdot^x$ is replaced by a reflected process $X_\cdot^x$ in $\overline{G}$, namely:

$$Y_t^x = Y_T^x + \int_t^T [\psi(X_s^x, Z_s^x) - \lambda]\mathrm{d}s - \int_t^T Z_s^x \mathrm{d}W_s, \ \forall 0 \leq t \leq T < +\infty. \tag{2.3}$$

For this purpose, we use a penalization method to construct a sequence of processes $X^{x,n}$ defined on the whole $\mathbb{R}^d$ and which converges to the reflected process X^x. More precisely, we denote by $(Y^{x,\alpha,n,\varepsilon}, Z^{x,\alpha,n,\varepsilon})$ the solution of the following BSDE with regularized coefficients ψ^ε, d^ε, F_n^ε and b^ε by convolution with a sequence approximating the identity, $\forall t, T \in \mathbb{R}_+, \quad 0 \leq t \leq T < +\infty$:

$$Y_t^{x,\alpha,n,\varepsilon} = Y_T^{x,\alpha,n,\varepsilon} + \int_t^T [\psi^\varepsilon(X_s^{x,n,\varepsilon}, Z_s^{x,\alpha,n,\varepsilon}) - \alpha Y_s^{x,\alpha,n,\varepsilon}]\mathrm{d}s - \int_t^T Z_s^{x,\alpha,n,\varepsilon}\mathrm{d}W_s, \tag{2.4}$$

where $X^{x,n,\varepsilon}$ is the strong solution of the SDE:

$$X_t^{x,n,\varepsilon} = x + \int_0^t (d^\varepsilon + F_n^\varepsilon + b^\varepsilon)(X_s^{x,n,\varepsilon})\mathrm{d}s + \int_0^t \sigma^\varepsilon(X_s^{x,n,\varepsilon})\mathrm{d}W_s.$$

Note that as F_n is dissipative with a dissipative constant equal to 0, $d + F_n$ remains dissipative with a dissipative coefficient equal to η. Then, making $\varepsilon \to 0$, $n \to +\infty$ and $\alpha \to 0$, it is possible to show that, roughly speaking, $(Y_t^{x,\alpha,n,\varepsilon} - Y_0^{x,\alpha,n,\varepsilon}, Z_t^{x,\alpha,n,\varepsilon}, Y_0^{x,\alpha,n,\varepsilon}) \to (Y_t^x, Z_t^x, \lambda)$ which is solution of EBSDE (2.3). Once a solution (Y, Z, λ) is found for the EBSDE (2.3) we study existence and uniqueness of solutions of the type (Y, Z, λ) and (Y, Z, μ) of the EBSDE (2.1). Here we only manage to find solutions which are not Markovian and which are not bounded in expectation. Then we show that the function defined by $v(x) := Y_0^x$, where Y is a solution of EBSDE (2.3) is a viscosity solution of the following partial differential equation (PDE in what follows) :

$$\begin{cases} \mathscr{L}v(x) + \psi(x, \nabla v(x)\sigma(x)) = \lambda, & x \in G, \\ \frac{\partial v}{\partial n}(x) = 0, & x \in \partial G, \end{cases} \tag{2.5}$$

where:

$$\mathscr{L}u(x) = \frac{1}{2}\operatorname{Tr}(\sigma(x){}^t\sigma(x)\nabla^2 u(x)) + {}^tf(x)\nabla u(x).$$

Note that the boundary ergodic problem

$$\begin{cases} F(D^2v, Dv, x) & = \lambda \text{ in } \overline{G} \\ L(Dv, x) & = \mu \end{cases}$$

were studied in [5] by Barles, Da Lio, Lions and Souganidis when $\overline{G}$ is a smooth, periodic, half-space-type domain and F a periodic function. They found a constant μ such that there exists a bounded viscosity solution v of the above problem.

At last we show that we can use the theory of EBSDE to solve an optimal ergodic control problem. $R : U \to \mathbb{R}^d$ is assumed to be bounded and L is assumed to be Lipschitz and bounded. We define the ergodic cost:

$$I(x, \rho) = \limsup_{T\to+\infty} \frac{1}{T}\mathbb{E}_T^\rho\left[\int_0^T L(X_s^x, \rho_s)\mathrm{d}s\right], \tag{2.6}$$

where ρ is an adapted process with values in a separable metric space U and $\mathbb{E}_T^\rho$ is the expectation with respect to the probability measure under which $W_t^\rho = W_t + \int_0^t R(\rho_s)\mathrm{d}s$ is a Brownian motion on $[0, T]$. Defining

$$\psi(x, z) = \inf_{u\in U}\{L(x, u) + zR(u)\}, \qquad x \in R^d, \quad z \in \mathbb{R}^{1\times d},$$

it is possible to show that, for any admissible control ρ, $I(x, \rho) \geq \lambda$. That is why λ is called ergodic cost. In a similar way, μ is called boundary ergodic cost.

The paper is organized as follows. In section 2, we study the forward SDE under the hypothesis that the drift is weakly dissipative and that the diffusion matrix is invertible and bounded. In this section we prove that the estimates we establish depend on d through its dissipativity coefficient. In section 3, we use the basic coupling estimate to study existence and uniqueness of an EBSDE with zero Neumann boundary conditions with a forward process weakly dissipative but non-reflected. In section 4, we use a penalization method to show that the same result holds for a reflected process in a convex set not necessarily bounded. Then, we establish the link between the EBSDE with zero Neumann boundary condition and a PDE. Finally, we apply our results to an optimal ergodic control problem. Some technical proofs are given in the Appendix.

2.2 The forward SDE

2.2.1 General notation

The canonical scalar product on $\mathbb{R}^d$ is denoted by $(\ ,\ )$ and the associated norm is denoted by $|\cdot|$. Given a matrix $\sigma \in \mathbb{R}^{d\times d}$, we define by $|||\cdot|||$ its operator norm. Let $\mathscr{O}$ be an open connected subset of $\mathbb{R}^d$. We denote by $\mathscr{C}_b^k(\mathscr{O})$ the set of real functions of class $\mathscr{C}^k$ on $\mathscr{O}$ with bounded partial derivatives. We denote by $\mathscr{C}_{\text{lip}}^k$ the set of real functions whose partial derivatives of order less than or equal to k are Lipschitz. We denote by $B_b(\mathscr{O})$ the set of Borel measurable bounded functions defined on $\mathscr{O}$.

$(\Omega, \mathscr{F}, \mathbb{P})$ denotes a complete probability space, $(W_t)_{t\geq 0}$ denotes an $\mathbb{R}^d$-valued standard Brownian motion defined on this space and $(\mathscr{F}_t)_{t\geq 0}$ is the natural filtration of W augmented by $\mathbb{P}$-null sets. Then $(\mathscr{F}_t)_{t\geq 0}$ satisfies the usual condition.

$\mathscr{S}^2$ denotes the space of real-valued adapted continuous processes Y such that for all $T > 0$, $\mathbb{E}[\sup_{0\leq t\leq T} |Y_t|^2] < +\infty$.

$\mathscr{S}^p$ denotes the space of real-valued adapted continuous processes Y such that for all $T > 0$, $\mathbb{E}[\sup_{0\leq t\leq T} |Y_t|^p] < +\infty$.

$\mathscr{M}^2(\mathbb{R}_+, \mathbb{R}^k)$ denotes the space consisting of all progressively measurable processes X, with value in $\mathbb{R}^k$ such that, for all $T > 0$,

$$\mathbb{E}\left[\int_0^T |X_s|^2 \mathrm{d}s\right] < +\infty.$$

Let $f : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ be two locally Lipschitz functions. We denote by $(V_t^x)_{t\geq 0}$ the strong solution of the following SDE:

$$V_t^x = x + \int_0^t f(V_s^x)\mathrm{d}s + \int_0^t \sigma(V_s^x)\mathrm{d}W_s. \tag{2.7}$$

Lemma 2.1. *Assume that $\exists a \in \mathbb{R}^d$, η_1, $\eta_2 > 0$ such that, $\forall y \in \mathbb{R}^d$,*

$$(f(y), y - a) \leq -\eta_1 |y - a|^2 + \eta_2,$$

and that $|\sigma|$ is bounded by σ_∞, then there exists a strong solution $(V_t^x)_{t\geq 0}$ to (2.7) which is pathwise unique and for which the explosion time is almost surely equal to infinity. Furthermore the following estimate holds $\forall t \geq 0$:

$$\mathbb{E}|V_t^x|^2 \leq C(1 + |x|^2 e^{-2\eta_1 t}),$$

where C is a constant which does not depend on time t and depends on f only through η_1 and η_2, and on σ only through σ_∞. Furthermore, for all $p > 2$, for all $0 < \beta < p\eta_1$,

$$\mathbb{E}|V_t^x|^p \leq C(1 + |x|^p e^{-\beta t}),$$

where C is a constant which does not depend on time t, depends on f only through η_1 and η_2, and on σ only through σ_∞. We also have the following inequality, for all $p \geq 1$,

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |V_t^x|^p\right] \leq C(1 + |x|^p),$$

where C depends in this case, on time T.

Proof. The proof is given in the appendix. □

We recall that a function is weakly dissipative if it is a sum of an η-dissipative function (namely $\forall x, x' \in \mathbb{R}^d$, $(d(x) - d(x'), x - x') \leq -\eta|x - x'|^2$), and a bounded function. Thus we write $f = d + b$, with d η-dissipative and $|b|$ bounded by B.

Hypothesis 2.1.

- $f = d + b$ is weakly dissipative,
- d is locally Lipschitz and have polynomial growth: there exists $\nu > 0$ such that for all $x \in \mathbb{R}^d$, $|d(x)| \leq C(1 + |x|^\nu)$,
- b is Lipschitz,
- σ is Lipschitz, invertible, and $|\sigma|$ and $|\sigma^{-1}|$ are bounded by σ_∞,

– there exists $\Lambda \geq 0$ such that for every $x, y \in \mathbb{R}^d$,

$$|\,(y, \sigma(x+y) - \sigma(x))\,| \leq \Lambda |y|$$

and there exists $\lambda > 0$ such that

$$2(\lambda - \lambda^2 \Lambda^2) > |||\sigma^{-1}|||^2.$$

Remark 2.2. Note that when $\sigma(x)$ does not depend on x, then $\Lambda = 0$ and the last assumption of Hypothesis 2.1 is satisfied for λ large enough. We give an example of σ depending on x and satisfying the last assumption. In the one dimensional case, take $\sigma(x) = 10\mathbb{1}_{\{x\leq 0\}} + (10 + \frac{1}{10}x)\mathbb{1}_{\{0<x<1\}} + (101/10)\mathbb{1}_{\{x\geq 1\}}$. Then $\Lambda = 1/10$ and $|||\sigma^{-1}||| \leq 1/10$. Clearly, the assumption is satisfied for $\lambda = 1$.

Remark 2.3. It is clear that if f satisfies Hypothesis 2.1 then f satisfies the assumption of Lemma 2.1. Indeed, let us suppose that f satisfies Hypothesis 2.1. Let $a \in \mathbb{R}^d$, then $\forall y \in \mathbb{R}^d$,

$$\begin{aligned}
&(f(y) - f(a), y - a) = (d(y) - d(a), y - a) + (b(y) - b(a), y - a) \\
\Rightarrow &(f(y), y - a) \leq -\eta |y - a|^2 + 2B|y - a| + |f(a)||y - a| \\
\Rightarrow &(f(y), y - a) \leq -\eta |y - a|^2 + \frac{(2B + |f(a)|)^2}{2\varepsilon} + \frac{\varepsilon |y - a|^2}{2},
\end{aligned}$$

which gives us the desired result, for ε small enough.

Lemma 2.4. *Assume that Hypothesis (2.1) holds true but this time with b replaced by b_2 which is only bounded measurable and not Lipschitz anymore. Then the solution of (2.7) with b replaced by b_2 still exists but in the weak sense, namely there exists a new Brownian motion $(\hat{W}_t)_{t\geq 0}$ with respect to a new probability measure $\hat{\mathbb{P}}$ under which equation (2.7) is satisfied by $(V_t^x)_{t\geq 0}$ with $(W_t)_{t\geq 0}$ replaced by $(\hat{W}_t)_{t\geq 0}$. Such a process is unique in law and the estimates of Lemma (2.1) are still satisfied under the new probability $\hat{\mathbb{P}}$.*

Proof. It is enough to write:

$$\begin{aligned}
\mathrm{d}V_t^x &= [d(V_t^x) + b(V_t^x)]\mathrm{d}t + \sigma(V_t^x)\mathrm{d}W_t \\
&= [d(V_t^x) + b_2(V_t^x)]\mathrm{d}t + \sigma(V_t^x)[\sigma^{-1}(V_t^x)(b(V_t^x) - b_2(V_t^x)) + \mathrm{d}W_t] \\
&= [d(V_t^x) + b_2(V_t^x)]\mathrm{d}t + \sigma(V_t^x)\mathrm{d}\hat{W}_t,
\end{aligned}$$

where $\mathrm{d}\hat{W}_t = \sigma^{-1}(V_t^x)(b(V_t^x) - b(V_t^x)) + \mathrm{d}W_t$ is the new Brownian motion thanks to the Girsanov theorem (note that σ^{-1}, b and b_2 are measurable and bounded by hypothesis). □

Lemma 2.5. *Assume that Hypothesis 2.1 holds true. Then there exist $C > 0$ and $\mu > 0$ such that $\forall \Phi \in B_b(\mathbb{R}^d)$,*

$$|\mathscr{P}_t[\Phi](x) - \mathscr{P}_t[\Phi](x')| \leq C(1 + |x|^2 + |x'|^2)e^{-\mu t}|\Phi|_0 \tag{2.8}$$

where $\mathscr{P}_t[\Phi](x) = \mathbb{E}\Phi(V_t^x)$ is the Kolmogorov semigroup associated to (2.7). We stress the fact that the constants C and μ depend on f only through η and B.

Proof. The proof is given in the appendix. □

Remark 2.6. The importance of the dependency of C and μ only through some parameters of the problem will appear in Remark 2.14.

Corollary 2.7. *The estimate (2.8) can be extended to the case in which b is only bounded measurable and there exists a uniformly bounded sequence of Lipschitz functions $\{b_m\}_{m\geq 1}$ (i.e. b_m is Lipschitz and $\sup_m \sup_x |b_m(x)| < +\infty$) such that*

$$\forall x \in \mathbb{R}^d, \quad \lim_m b_m(x) = b(x).$$

In this case, we define a semigroup relatively to the new probability measure, namely:

$$\mathscr{P}_t[\Phi](x) := \hat{\mathbb{E}}\Phi(X_t^x).$$

Proof. We denote by $\mathscr{P}_t^m$ the Kolmogorov semigroup of (2.7) with b replaced by b_m, for more clarity we rewrite this equation below: $\forall x \in \overline{G}$,

$$V_t^{x,m} = x + \int_0^t (d + b_m)(V_s^{x,m})\mathrm{d}s + \int_0^t \sigma(V_s^{x,m})\mathrm{d}W_s.$$

It is sufficient to prove that, $\forall x \in \overline{G}, \ \forall t \geq 0$,

$$\mathscr{P}_t^m[\Phi](x) \to \mathscr{P}_t[\Phi](x).$$

To do that, it is easy to adapt the proof from [26] replacing the process U_t^x by its analogue in our context. Thus we define U_t^x as the strong solution of the following SDE:

$$U_t^x = x + \int_0^t d(U_s^x)\mathrm{d}s + \int_0^t \sigma(U_s^x)\mathrm{d}W_s,$$

and the rest remains the same. □

2.3 The ergodic BSDE

In this section we study the following EBSDE in infinite horizon:

$$Y_t^x = Y_T^x + \int_t^T [\psi(V_s^x, Z_s^x) - \lambda]\mathrm{d}s - \int_t^T Z_s^x \mathrm{d}W_s, \ \forall 0 \leq t \leq T < +\infty. \tag{2.9}$$

At the moment, the forward process, defined as the strong solution of (2.7) is not reflected. However the existence result we are going to show in the next theorem is interesting for its own, because it gives some ideas which will be reused in the next section.

We need the following hypothesis on $\psi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$:

Hypothesis 2.2. There exists $M_\psi \in \mathbb{R}$ such that: $\forall x \in \mathbb{R}^d$, $\forall z, \, z' \in \mathbb{R}^{1\times d}$,
- $\psi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is measurable,
- $\psi(\cdot, z)$ is continuous ,
- $|\psi(x, 0)| \leq M_\psi$,
- $|\psi(x, z) - \psi(x, z')| \leq M_\psi |z - z'|$.

Hypothesis 2.3.
- f is $\mathscr{C}^1$ and all of its derivatives have polynomial growth of first order, i.e. for each $x \in \mathbb{R}^d$ and each multi-index L with $|L| \leq m$, $m \in \{0, 1\}$, there exist positive constants γ_m and q_m such that

  $$|\partial_L d(x)|^2 \leq \gamma_m(1 + |x|^{q_m}).$$

  Also, set $\xi := \max_{m\in\{0,1\}} q_m < +\infty$.
- b, σ and $\psi \ \in \ \mathscr{C}_b^1$.

Using a standard approach (see [31]), we are going to study the following BSDE in infinite horizon

$$Y_t^{x,\alpha} = Y_T^{x,\alpha} + \int_t^T [\psi(V_s^x, Z_s^{x,\alpha}) - \alpha Y_s^{x,\alpha}]\mathrm{d}s - \int_t^T Z_s^{x,\alpha}\mathrm{d}W_s, \ \forall 0 \leq t \leq T < +\infty. \quad (2.10)$$

Such an equation was studied in [11] from which we have the following result:

Lemma 2.8. *Assume that hypotheses (2.1) and (2.2) hold true. Then there exists a unique solution $(Y^{x,\alpha}, Z^{x,\alpha})$ to BSDE (2.10) such that $Y^{x,\alpha}$ is a bounded adapted continuous process and $Z^{x,\alpha} \in \mathscr{M}^2(\mathbb{R}_+, \mathbb{R}^{1\times d})$. Furthermore, $|Y_t^{x,\alpha}| \leq \frac{M_\psi}{\alpha}$. Finally there exists a function v^α such that $Y_t^{x,\alpha} = v^\alpha(X_t^x)$ $\mathbb{P}$-a.s. and there exists a measurable function $\zeta^\alpha : \mathbb{R}^d \to \mathbb{R}^{1\times d}$ such that $Z_t^{x,\alpha} = \zeta^\alpha(X_t^x)$ $\mathbb{P}$-a.s.*

We will need the following lemma :

Lemma 2.9. *Let ζ, ζ' be two continuous functions: $\mathbb{R}^d \to \mathbb{R}^{1\times d}$. We define*

$$\Upsilon(x) = \begin{cases} \frac{\psi(x,\zeta(x)) - \psi(x,\zeta'(x))}{|\zeta(x)-\zeta'(x)|^2}{}^t(\zeta(x) - \zeta'(x)), \ \text{if } \zeta(x) = \zeta'(x), \\ 0, \ \text{if } \zeta(x) = \zeta'(x). \end{cases}$$

There exists a uniformly bounded sequence of Lipschitz functions $(\Upsilon_n)_{n\geq 0}$ (i.e., $\forall n$, Υ_n is Lipschitz and $\sup_n \sup_x |\Upsilon_n(x)| < +\infty$) such that Υ_n converges pointwisely to Υ.

Proof. For all $n \in \mathbb{N}$, we fix infinitely differentiable functions $\rho_n : \mathbb{R}^d \to \mathbb{R}_+$ bounded together with their derivatives of all order, such that $\int_{\mathbb{R}^d} \rho_n(x)\mathrm{d}x = 1$ and

$$\operatorname{supp}(\rho_n) \subset \left\{ x \in \mathbb{R}^d, |x| \leq \frac{1}{n} \right\},$$

where supp denotes the support. Then the required functions Υ can be defined as $\Upsilon_n(x) := \int_{\mathbb{R}^d} \rho_n(x-y)\Upsilon(y)\mathrm{d}y$. □

The following lemma gives us the desired estimates on $v^\alpha(x)$ which will allow us to apply a diagonal procedure.

Lemma 2.10. *Assume that the Hypotheses (2.1), (2.2) and (2.3) hold true. Then, there exists a constant $C > 0$ independent of α and which depends on f only through η and B, on σ only through σ_∞ and on ψ only through M_ψ such that, $\forall x, y \in \mathbb{R}^d$,*

$$|v^\alpha(x) - v^\alpha(y)| \leq C(1 + |x|^2 + |y|^2),$$
$$|v^\alpha(x) - v^\alpha(y)| \leq C(1 + |x|^2 + |y|^2)|x-y|.$$

Proof. Let us introduce, as in the paper [76], equation (3.4), some smooth functions $\Phi_m : \mathbb{R}^d \to \mathbb{R}$ such that defining

$$d_m(x) = \Phi_m(x)d(x), \forall x \in \mathbb{R}^d,$$

d_m is globally Lipschitz, continuously differentiable and for each multi-index L with $|L| \leq 1$,

$$\sup_{m,x} \{|\partial_L \phi_m(x)| + |d(x)\partial_L \phi_m(x)|\} \leq C, \quad (2.11)$$

for some $C > 0$. We recall that $\phi_m = 1$ on $A_m := \{x \in \mathbb{R}^d; |x| \leq m^\xi\}$. Furthermore,

$$\nabla d_m(x) = \begin{pmatrix} \frac{\partial \phi_m(x)}{\partial x_1} d_1(x) & \dots & \frac{\partial \phi_m(x)}{\partial x_d} d_1(x) \\ \vdots & & \vdots \\ \frac{\partial \phi_m(x)}{\partial x_1} d_d(x) & \dots & \frac{\partial \phi_m(x)}{\partial x_d} d_d(x) \end{pmatrix} + \phi_m(x) \nabla d(x)$$
$$= d(x) \nabla \phi_m(x) + \phi_m(x) \nabla d(x).$$

Now let us consider $(V_t^{x,m})_{t \geq 0}$ the unique solution of

$$V_t^{x,m} = x + \int_0^t [d_m(V_s^{x,m}) + b(V_s^{x,m})] \mathrm{d}s + \int_0^t \sigma(V_s^{x,m}) \mathrm{d}W_s, \quad \forall t \geq 0. \tag{2.12}$$

We recall that, for each $t \geq 0$ and $p > 1$, $V_t^{x,m}$ converges to V_t in L^p and almost surely. Furthermore the following estimates hold, thanks to the proof of Lemma 3.2 in [76], for every $p \in \mathbb{N} \setminus \{0, 1\}$, there exists $C_p > 0$ such that

$$\sup_{m \geq 1} \sup_{0 \leq t \leq T} \mathbb{E}\left[|V_t^{x,m}|^p\right] \leq C_p(1 + |x|^{p\nu}) e^{C_p T}. \tag{2.13}$$

We denote by $\nabla V_s^{x,m} = (\nabla_1 V_s^{x,m}, \dots, \nabla_d V_s^{x,m})$ the solution of the following variational equation (see equation (2.9) in [54]):

$$\nabla_i V_s^{x,m} = e_i + \int_t^s \nabla(d_m + b)(V_r^{x,m}) \nabla_i V_r^{x,m} \mathrm{d}r + \sum_{j=1}^d \int_t^s [\nabla \sigma^j(V_r^{x,m})] \nabla_i V_r^{x,m} \mathrm{d}W_r^j.$$

Let us mention that the following estimate holds, for every $p > 1$:

$$\sup_{m \geq 1} \sup_{0 \leq t \leq T} \mathbb{E}\left[|\nabla V_t^{x,m}|^p\right] \leq C_{p,T}. \tag{2.14}$$

Let us denote by $(Y_t^{x,\alpha,m}, Z_t^{x,\alpha,m})_{t \geq 0}$ the unique solution of the following monotone BSDE in infinite horizon, $\forall 0 \leq t \leq T < +\infty$,

$$Y_t^{x,\alpha,m} = Y_T^{x,\alpha,m} + \int_t^T [\psi(V_s^{x,m}, Z_s^{x,\alpha,m}) - \alpha Y_s^{x,\alpha,m}] \mathrm{d}s - \int_t^T Z_s^{x,\alpha,m} \mathrm{d}W_s.$$

We recall that if we denote by $v^{\alpha,m}$ the following quantity:

$$v^{\alpha,m}(x) := Y_0^{x,\alpha,m}$$

then $Y_s^{x,\alpha,m} = v^{\alpha,m}(V_s^x)$.

Now remark that Theorem 4.2 in [54] asks for the terminal condition of the BSDE to be Lipschitz, whereas in our case, the terminal condition $v^{\alpha,m}$ is only continuous and bounded. However, using the same method as that in Theorem 4.2 in [32], we can readily extend Theorem 4.2 in [54] for a Markovian terminal condition which is only continuous with polynomial growth, which is our case here. So, by Theorem 4.2 in [54], $v^{\alpha,m}$ is continuously differentiable,

$$Z_s^{x,\alpha,m} = \nabla v^{\alpha,m}(V_s^{x,m}) \sigma(V_s^{x,m}), \qquad \forall s \in [0, T], \qquad \mathbb{P}\text{-a.s},$$

and

$$\nabla v^{\alpha,m}(x) = \mathbb{E}\left[v^\alpha(V_T^{x,m}) N_T^{0,m} + \int_0^T [\psi(V_r^{x,m}, Z_r^{x,\alpha,m}) - \alpha Y_r^{x,\alpha,m}] \, N_r^{0,m} \mathrm{d}r\right],$$

where

$$N_r^{0,m} = \frac{1}{r}{}^t\left(\int_0^r {}^t[\sigma^{-1}(V_s^{x,m})\nabla V_s^{x,m}]\mathrm{d}W_s\right).$$

We immediately deduce the following estimate

$$\begin{aligned}\mathbb{E}|N_r^{0,m}|^2 &= \frac{1}{|r|^2}\mathbb{E}\int_0^r |\sigma^{-1}(V_s^{x,m})\nabla V_s^{x,m}|^2\mathrm{d}s \\ &\leq \frac{1}{|r|^2}\sigma_\infty^2\int_0^r \mathbb{E}|\nabla V_s^{x,m}|^2\mathrm{d}s \\ &\leq \frac{C_T}{r}. \end{aligned}\tag{2.15}$$

As a consequence we can immediately get a uniform bound in m for $\nabla v^{\alpha,m}(x)$. Indeed,

$$\begin{aligned}|\nabla v^{\alpha,m}(x)| &\leq \frac{M_\psi}{\alpha}\sqrt{\mathbb{E}\left(\left|N_T^{0,m}\right|^2\right)} + \int_0^T \sqrt{\mathbb{E}\left[(2M_\psi + M_\psi|Z_s^{x,\alpha,m}|)^2\right]}\sqrt{\mathbb{E}\left(\left|N_r^{0,m}\right|^2\right)}\mathrm{d}r \\ &\leq \frac{M_\psi}{\alpha}\frac{C_T}{\sqrt{T}} + \int_0^T C\left(1+\sqrt{\mathbb{E}|\nabla v^{\alpha,m}(V_r^{x,m})|^2}\right)\frac{1}{\sqrt{r}}\mathrm{d}r.\end{aligned}$$

We define $|||\nabla v^{\alpha,m}(x)||| := \sup_{x\in\mathbb{R}^d}|\nabla v^{\alpha,m}(x)|$, then,

$$|\nabla v^{\alpha,m}(x)| \leq \frac{C_{T,\alpha}}{\sqrt{T}} + C\sqrt{T}(1+|||\nabla v^{\alpha,m}|||),$$

which implies that, taking the supremum over x and for T small enough, for all $x\in\mathbb{R}^d$,

$$|\nabla v^{\alpha,m}(x)| \leq C_{T,\alpha}.\tag{2.16}$$

Now we claim that for every $T \geq 0$,

$$\mathbb{E}\left[|Y_T^{x,\alpha,m} - Y_T^{x,\alpha}|^2\right] + \mathbb{E}\int_0^T |Z_s^{x,\alpha,m} - Z_s^{x,\alpha}|^2\mathrm{d}s \underset{m\to+\infty}{\longrightarrow} 0.$$

For that purpose, let us denote by $(Y_t^{x,\alpha,n}, Z_t^{x,\alpha,n})$ the solution of the following finite horizon BSDE, for all $0\leq t\leq n$,

$$Y_t^{x,\alpha,n} = 0 + \int_t^n \left[\psi(V_s^x, Z_s^{x,\alpha,n}) - \alpha Y_s^{x,\alpha,n}\right]\mathrm{d}s - \int_t^n Z_s^{x,\alpha,n}\mathrm{d}W_s.$$

By inequality (12) in [11], $\mathbb{P}$-a.s., for all $0\leq t\leq n$,

$$\mathbb{E}|Y_t^{x,\alpha,n} - Y_t^{x,\alpha}|^2 + \mathbb{E}\int_0^t |Z_t^{x,\alpha,n} - Z_t^{x,\alpha}|^2\mathrm{d}s \leq Ce^{-2\alpha n},$$

where C depends only on M_ψ and α. Similarly, let us denote by $(Y_t^{x,\alpha,m,n}, Z_t^{x,\alpha,m,n})$ the solution of the following finite horizon BSDE, for all $0\leq t\leq n$,

$$Y_t^{x,\alpha,m,n} = 0 + \int_t^n \left[\psi(V_s^{x,m}, Z_s^{x,\alpha,m,n}) - \alpha Y_s^{x,\alpha,m,n}\right]\mathrm{d}s - \int_t^n Z_s^{x,\alpha,m,n}\mathrm{d}W_s.$$

Again, by inequality (12) in [11], $\mathbb{P}$-a.s., for all $0\leq t\leq n$,

$$\mathbb{E}|Y_t^{x,\alpha,m,,n} - Y_t^{x,\alpha,m}|^2 + \mathbb{E}\int_0^t |Z_t^{x,\alpha,m,n} - Z_t^{x,\alpha,m}|^2\mathrm{d}s \leq Ce^{-2\alpha n},$$

where C depends only on M_ψ and α.

Furthermore, thanks to the continuity of ψ in x, the following stability result for BSDEs in finite horizon holds (see for example Lemma 2.3 in [11]), for all $0 \leq t \leq n$:

$$\mathbb{E}|Y_t^{x,\alpha,m,n} - Y_t^{x,\alpha,n}|^2 + \mathbb{E}\int_0^n |Z_t^{x,\alpha,m,n} - Z_t^{x,\alpha,n}|^2 \mathrm{d}s \underset{m\to\infty}{\longrightarrow} 0.$$

Then,

$$\begin{aligned}
\mathbb{E}\left[|Y_T^{x,\alpha,m} - Y_T^{x,\alpha}\ |^2\right] &+ \mathbb{E}\int_0^T |Z_s^{x,\alpha,m} - Z_s^{x,\alpha}|^2 \mathrm{d}s \\
&\leq 3\mathbb{E}\left[|Y_T^{x,\alpha,m} - Y_T^{x,\alpha,m,n}|^2\right] + 3\mathbb{E}\left[|Y_T^{x,\alpha,m,n} - Y_T^{x,\alpha,n}|^2\right] \\
&\quad + 3\mathbb{E}\left[|Y_T^{x,\alpha,n} - Y_T^{x,\alpha}|^2\right] + 3\mathbb{E}\int_0^T |Z_s^{x,\alpha,m} - Z_s^{x,\alpha,m,n}|^2 \mathrm{d}s \\
&\quad + 3\mathbb{E}\int_0^T |Z_s^{x,\alpha,m,n} - Z_s^{x,\alpha,n}|^2 \mathrm{d}s + 3\mathbb{E}\int_0^T |Z_s^{x,\alpha,n} - Z_s^{x,\alpha}|^2 \mathrm{d}s \\
&\leq Ce^{-2\alpha n} + 3\mathbb{E}\left[|Y_T^{x,\alpha,m,n} - Y_T^{x,\alpha,n}|^2\right] + 3\mathbb{E}\int_0^T |Z_s^{x,\alpha,m,n} - Z_s^{x,\alpha,n}|^2 \mathrm{d}s.
\end{aligned}$$

Now, for every $\varepsilon > 0$, we pick n large enough such that $2\frac{M_\psi}{\alpha}e^{-\alpha n} < \varepsilon/2$. Then, we choose m large enough such that $3\mathbb{E}\left[|Y_T^{x,\alpha,m,n} - Y_T^{x,\alpha,n}|^2\right] + 3\mathbb{E}\int_0^T |Z_s^{x,\alpha,m,n} - Z_s^{x,\alpha,n}|^2 \mathrm{d}s < \varepsilon/2$. This shows that, for all $x \in \mathbb{R}^d$ and $T \geq 0$,

$$\mathbb{E}\left[|Y_T^{x,\alpha,m} - Y_T^{x,\alpha}|^2\right] + \mathbb{E}\int_0^T |Z_s^{x,\alpha,m} - Z_s^{x,\alpha}|^2 \mathrm{d}s \underset{m\to+\infty}{\longrightarrow} 0. \tag{2.17}$$

In particular taking $T = 0$ we deduce that for every $x \in \mathbb{R}^d$,

$$\lim_{m\to+\infty} v^{\alpha,m}(x) = v^\alpha(x).$$

Therefore, if we show that $\lim_{m\to+\infty} \nabla v^{\alpha,m}(x) = h^\alpha(x)$ for some function $h^\alpha(x)$ then this will imply that v^α is continuously differentiable and $\nabla v^\alpha = h^\alpha$. Furthermore, as $\mathbb{E}\int_0^T |Z_s^{x,\alpha,m} - Z_s^{x,\alpha}|^2 \mathrm{d}s \underset{m\to+\infty}{\longrightarrow} 0$ and $Z_s^{x,\alpha,m} = \nabla v^{\alpha,m}(V_s^{x,m})\sigma(V_s^{x,m})$, then it will imply that for a.a. $s \geq 0$, $\mathbb{P}$-a.s.,

$$Z_s^{x,\alpha} = \nabla v^\alpha(V_s^x)\sigma(V_s^x). \tag{2.18}$$

Now we claim that h^α can be written as

$$h^\alpha(x) = \mathbb{E}\left[v^\alpha(V_T^x)N_T^0 + \int_0^T \left[\psi(V_r^x, Z_r^{x,\alpha}) - \alpha Y_r^{x,\alpha}\right] N_r^0 \mathrm{d}r\right],$$

where

$$N_r^0 = \frac{1}{r}{}^t\left(\int_0^r {}^t[\sigma^{-1}(V_s^x)\nabla V_s^x]\mathrm{d}W_s\right).$$

Indeed, for every $x \in \mathbb{R}^d$,

$$\begin{aligned}
&|\nabla v^{\alpha,m}(x) - h^\alpha(x)| \\
&\quad \leq \mathbb{E}|v^{\alpha,m}(V_T^{x,m})N_T^{0,m} - v^\alpha(V_T^x)N_T^0| \\
&\qquad + \mathbb{E}\int_0^T |(\psi(V_r^{x,m}, Z_r^{x,\alpha,m}) - \alpha Y_r^{x,\alpha,m})N_r^{0,m} - (\psi(V_r^x, Z_r^{x,\alpha}) - \alpha Y_r^{x,\alpha})N_r^0|\mathrm{d}r \\
&\quad \leq \mathbb{E}|v^{\alpha,m}(V_T^{x,m})N_T^{0,m} - v^{\alpha,m}(V_T^{x,m})N_T^0| + \mathbb{E}|v^{\alpha,m}(V_T^{x,m})N_T^0 - v^\alpha(V_T^x)N_T^0| \\
&\qquad + \mathbb{E}\int_0^T |(\psi(V_r^{x,m}, Z_r^{x,\alpha,m}) - \alpha Y_r^{x,\alpha,m})N_r^{0,m} - (\psi(V_r^{x,m}, Z_r^{x,\alpha,m}) - \alpha Y_r^{x,\alpha,m})N_r^0|\mathrm{d}r \\
&\qquad + \mathbb{E}\int_0^T |(\psi(V_r^{x,m}, Z_r^{x,\alpha,m}) - \alpha Y_r^{x,\alpha,m})N_r^0 - (\psi(V_r^x, Z_r^{x,\alpha}) - \alpha Y_r^{x,\alpha})N_r^0|\mathrm{d}r \\
&\quad \leq \frac{M_\psi}{\alpha}\sqrt{\mathbb{E}(|N_T^{0,m} - N_T^0|^2)} + \sqrt{\mathbb{E}(|N_T^0|^2)}\sqrt{\mathbb{E}(|v^{\alpha,m}(V_T^{x,m}) - v^\alpha(V_T^x)|^2)} \\
&\qquad + \int_0^T \sqrt{C(1 + \mathbb{E}(|\nabla v^{\alpha,m}(V_s^{x,m})|^2))}\sqrt{\mathbb{E}(|N_r^{0,m} - N_r^0|^2)}\mathrm{d}r \\
&\qquad + \int_0^T \sqrt{\mathbb{E}(|N_r^0|^2)}\sqrt{\mathbb{E}(|\psi(V_r^{x,m}, Z_r^{x,\alpha,m}) - \alpha Y_r^{x,\alpha,m} - \psi(V_r^x, Z_r^{x,\alpha}) + \alpha Y_r^{x,\alpha}|^2)}\mathrm{d}r \\
&\quad \leq \frac{M_\psi}{\alpha}\sqrt{\mathbb{E}(|N_T^{0,m} - N_T^0|^2)} + \sqrt{\mathbb{E}(|N_T^0|^2)}\sqrt{\mathbb{E}(|v^{\alpha,m}(V_T^{x,m}) - v^\alpha(V_T^x)|^2)} \\
&\qquad + C_{T,\alpha}(\sqrt{T} + \frac{1}{\sqrt{T}})\int_0^T \sqrt{\mathbb{E}(|N_r^{0,m} - N_r^0|^2)}\mathrm{d}r \\
&\qquad + \int_0^T \frac{C_T}{\sqrt{r}} C_{T,\alpha}\sqrt{\mathbb{E}(|Z_r^{x,\alpha,m} - Z_r^{x,\alpha}|^2)}\mathrm{d}r \\
&\qquad + \sqrt{\mathbb{E}(|\psi(V_r^{x,m}, Z_r^{x,\alpha}) - \psi(V_r^x, Z_r^{x,\alpha})|^2)}\mathrm{d}r \\
&\qquad + \int_0^T \frac{C_T}{\sqrt{r}}\sqrt{\mathbb{E}(|Y_r^{x,\alpha,m} - Y_r^{x,\alpha}|^2)}\mathrm{d}r
\end{aligned}$$

where we have used the estimate (2.16) for the last inequality.

We have

$$\mathbb{E}(|N_T^{0,m} - N_T^0|^2) = \frac{1}{T^2}\int_0^T \mathbb{E}(|\sigma^{-1}\nabla V_s^{x,m} - \sigma^{-1}\nabla V_s^x|^2)\mathrm{d}s \underset{m\to+\infty}{\longrightarrow} 0,$$

since $\sigma^{-1}(V_s^{x,m})\nabla V_s^{x,m} - \sigma^{-1}(V_s^x)\nabla V_s^x \underset{m\to+\infty}{\longrightarrow} 0$ $\mathbb{P}$-a.s. and since

$$\sup_m \mathbb{E}(|\sigma^{-1}(V_s^{x,m})\nabla V_s^{x,m} - \sigma^{-1}(V_s^x)\nabla V_s^x|^4) < +\infty$$

by estimate (2.14).

The second and the third term in the sum converge toward 0 by the dominated convergence theorem. The fourth one converges toward 0 by Jensen's inequality and the dominated convergence theorem and the last two ones converge toward 0 by the dominated convergence theorem.

Now the first estimate of the lemma can be established exactly as in Lemma 3.6 of [26] thanks to the representation formula (2.18).

Let us establish the second inequality of the lemma. We have, using the following notation $\overline{v}^\alpha(x) = v^\alpha(x) - v^\alpha(0)$,

$$|\nabla\overline{v}^\alpha(x)| = \left|\mathbb{E}\left[\overline{v}^\alpha(V_T^x)N_T^0 + \int_0^T \left[\psi(V_s^x, Z_s^x) - \alpha\overline{v}^\alpha(V_r^x) - \alpha v^\alpha(0)\right]N_r^0\mathrm{d}r\right]\right|. \quad (2.19)$$

We have, using the first inequality of the lemma and inequality (2.15):

$$\mathbb{E}|\overline{v}^\alpha(V_T^x)N_T^0| \leq C\frac{(1+|x|^2)}{\sqrt{T}}.$$

Furthermore, since we can assume that $\alpha \leq 1$:

$$\begin{aligned}
&\mathbb{E}\int_0^T \left|\left[\psi(V_r^x, Z_r^{x,\alpha}) - \alpha\overline{Y}_r^{x,\alpha} - \alpha v^\alpha(0)\right] N_r^0\right| \mathrm{d}r \\
&\qquad \leq C\mathbb{E}\int_0^T \left(M_\psi + M_\psi|Z_r^{x,\alpha}| + C(1+|V_r^x|^2) + M_\psi\right)|N_r^0|\mathrm{d}r \\
&\qquad \leq C\mathbb{E}\int_0^T |N_r^0|\mathrm{d}r + C\mathbb{E}\int_0^T |Z_r^{x,\alpha}||N_r^0|\mathrm{d}r + C\mathbb{E}\int_0^T (1+|V_r^x|^2)|N_r^0|\mathrm{d}r \\
&\qquad =: I_1 + I_2 + I_3.
\end{aligned}$$

We easily get $I_1 \leq C$. Furthermore, thanks to the representation formula (2.18) and the fact that $|\sigma|$ is bounded:

$$I_2 \leq C\int_0^T \sqrt{\mathbb{E}|\nabla v^\alpha(V_s^x)|^2}\frac{1}{\sqrt{r}}\mathrm{d}r.$$

Regarding I_3, we easily get $I_3 \leq C(1+|x|^2)$.
Now we define $|||\nabla v^\alpha||| := \sup_{x\in\mathbb{R}^d} \frac{|\nabla v^\alpha(x)|}{1+|x|^2}$, then coming back to equation (2.19), we have

$$\begin{aligned}
|\nabla v^\alpha(x)| &\leq C\left(1+|x|^2+\frac{1+|x|^2}{\sqrt{T}}\right) + C\int_0^T \sqrt{\mathbb{E}(1+|V_s^x|^2)^2}|||\nabla v^\alpha|||\frac{1}{\sqrt{r}}\mathrm{d}r \\
&\leq C\left(1+|x|^2+\frac{1+|x|^2}{\sqrt{T}}\right) + C\int_0^T (1+|x|^2)|||\nabla v^\alpha|||\frac{1}{\sqrt{r}}\mathrm{d}r.
\end{aligned}$$

This implies

$$|||\nabla v^\alpha||| \leq C\left(1+\frac{1}{\sqrt{T}}\right) + C\sqrt{T}|||\nabla v^\alpha||||.$$

Thus, for T small enough:

$$|||\nabla v^\alpha||| \leq C\left(1+\frac{1+\sqrt{T}}{\sqrt{T}(1-C\sqrt{T})}\right),$$

which implies that, for all $x \in \mathbb{R}^d$,

$$|\nabla v^\alpha(x)| \leq C(1+|x|^2).$$

This last estimate gives us, for all $x, y \in \mathbb{R}^d$,

$$|v^\alpha(x) - v^\alpha(y)| \leq C(1+|x|^2+|y|^2)|x-y|.$$

□

Thanks to this estimate, it is possible to get an existence result for EBSDE (2.9). Here Hypothesis 2.3 can be removed thanks to a convolution argument which will appear in the proof.

Theorem 2.11. *Assume that the hypotheses (2.1) and (2.2) hold true. Then there exists a solution* $(\overline{Y}^x, \overline{Z}^x, \overline{\lambda})$ *to EBSDE (2.9) such that* $\overline{Y}^x_\cdot = \overline{v}(V^x_\cdot)$ *with* $\overline{v}$ *locally Lipschitz, and there exists a measurable function* $\overline{\xi} : \mathbb{R}^d \to \mathbb{R}^{1\times d}$ *such that* $\overline{Z}^x \in \mathscr{M}^2(\mathbb{R}_+, \mathbb{R}^{1\times d})$ *and* $\overline{Z}^x_\cdot = \overline{\xi}(V^x_\cdot)$.

Proof. We start by regularizing f and σ thanks to classical convolution arguments. For all $k \in \mathbb{N}^*$ let us denote by $\rho^k_\varepsilon : \mathbb{R}^k \to \mathbb{R}_+$ the classical mollifier for which the support is the ball of center 0 and radius ε. Let us denote for a sequence $(\varepsilon_n)_{n\in\mathbb{N}} \in \mathbb{R}_+$ such that $\varepsilon_n \underset{n\to+\infty}{\longrightarrow} 0$, $d^{\varepsilon_n} := d * \rho^d_{\varepsilon_n}$, $b^{\varepsilon_n} := b * \rho^d_{\varepsilon_n}$, and $\sigma^{\varepsilon_n} := \sigma * \rho^{d\times d}_{\varepsilon_n}$. Those functions are $\mathscr{C}^1$ and satisfies:

- d^{ε_n} is η-dissipative;
- $|d^{\varepsilon_n}(x)| \leq C(1+|x|)^p$, for a $p \geq 0$;
- $|\nabla d^{\varepsilon}(x)| \leq C_\varepsilon(1+|x|^q)$, for a $q \geq 0$;
- b^{ε_n} is bounded by B;
- σ^{ε_n} is invertible;
- $d^{\varepsilon_n} \to d$, $b^{\varepsilon_n} \to b$, $\sigma^{\varepsilon_n} \to \sigma$ pointwisely as $\varepsilon_n \to 0$.

Note now that Hypothesis 2.3 is satisfied by the regularized functions defined above, therefore Lemma 2.10 can be applied. We just precise that the pointwise convergence of the regularized functions is a consequence of the continuity of the functions d, b, and σ. We denote by V^{x,ε_n}_t the solution of (2.7) with f replaced by f^{ε_n} and σ replaced by σ^{ε_n}. The same notation is used for the regularized BSDE, we denote by $(Y^{x,\alpha,\varepsilon_n}_t, Z^{x,\alpha,\varepsilon_n}_t)$ the solution in $\mathscr{S}^2 \times \mathscr{M}^2(\mathbb{R}_+, \mathbb{R}^{1\times d})$ of BSDE (2.10) with V^x replaced by $V^{x,\varepsilon,n}$ (existence and uniqueness of such a solution is guaranteed by Lemma 2.8), namely $\forall 0 \leq t \leq T < +\infty$:

$$Y^{x,\alpha,\varepsilon_n}_t = Y^{x,\alpha,\varepsilon_n}_T + \int_t^T (\psi(V^{x,\varepsilon_n}_s, Z^{x,\alpha,\varepsilon_n}_s) - \alpha Y^{x,\alpha,\varepsilon_n}_s)\mathrm{d}s - \int_t^T Z^{x,\alpha,\varepsilon_n}_s \mathrm{d}W_s. \tag{2.20}$$

Then we define $v^{\alpha,\varepsilon_n}(x) := Y^{x,\alpha,\varepsilon_n}_0$ and $\overline{Y}^{x,\alpha,\varepsilon_n}_t = Y^{x,\alpha,\varepsilon_n}_t - \alpha v^{\alpha,\varepsilon_n}(0)$. We can rewrite the BSDE and we get:

$$\begin{aligned}\overline{Y}^{x,\alpha,\varepsilon_n}_t =& \overline{Y}^{x,\alpha,\varepsilon_n}_T + \int_t^T (\psi(V^{x,\varepsilon_n}_s, Z^{x,\alpha,\varepsilon_n}_s) - \alpha \overline{Y}^{x,\alpha,\varepsilon_n}_s - \alpha v^{\alpha,\varepsilon_n}(0))\mathrm{d}s \\ &- \int_t^T Z^{x,\alpha,\varepsilon_n}_s \mathrm{d}W_s, \quad 0 \leq t \leq T < +\infty.\end{aligned}$$

Uniqueness of solutions implies that $v^{\alpha,\varepsilon_n}(V^{x,\varepsilon_n}_s) = Y^{x,\alpha,\varepsilon_n}_s$. Now, in a very classical way, we set $\overline{v}^{\alpha,\varepsilon_n}(x) = v^{\alpha,\varepsilon_n}(x) - v^{\alpha,\varepsilon_n}(0)$. Thanks to the fact that $\alpha|v^{\alpha,\varepsilon_n}(0)| \leq M_\psi$ and by Lemma 2.10 we can extract a subsequence $\beta(\varepsilon_n) \underset{n\to+\infty}{\to} 0$ such that $\forall \alpha > 0$, $\forall x \in D$ a countable subset of $\mathbb{R}^d$:

$$\overline{v}^{\alpha,\beta(\varepsilon_n)}(x) \underset{n\to+\infty}{\longrightarrow} \overline{v}^\alpha(x) \quad \text{and} \quad \alpha v^{\alpha,\beta(\varepsilon_n)}(0) \underset{n\to+\infty}{\longrightarrow} \overline{\lambda}^\alpha,$$

for a suitable function $\overline{v}$ and a suitable real $\overline{\lambda}^\alpha$. Now thanks to the estimates from Lemma 2.10 we have $\forall \alpha > 0$, $|\overline{v}^{\alpha,\beta(\varepsilon_n)}(x) - \overline{v}^{\alpha,\beta(\varepsilon_n)}(x')| \leq c(1+|x|^2+|x'|^2)|x-x'|$ for all $x, x' \in \mathbb{R}^d$. Therefore extending $\overline{v}^\alpha$ to the whole $\mathbb{R}^d$ by setting $\overline{v}^\alpha(x) = \lim_{x_p\to x} \overline{v}^\alpha(x_p)$ we still have the following estimates: for all $x, x' \in \mathbb{R}^d$,

$$|\overline{v}^\alpha(x) - \overline{v}^\alpha(x')| \leq C(1+|x|^2+|x'|^2)|x-x'|.$$

In addition, we also have

$$|\overline{\lambda}^\alpha| \leq M_\psi.$$

Now let us define $\forall t \geq 0$, $\overline{Y}_t^{x,\alpha} = \overline{v}^\alpha(V_t^x)$. Let us show that

$$\mathbb{E}\int_0^T |\overline{Y}_s^{x,\alpha,\beta(\varepsilon_n)} - \overline{Y}_s^{x,\alpha}|^2 \mathrm{d}s \underset{n\to+\infty}{\to} 0 \text{ and } \mathbb{E}|\overline{Y}_T^{x,\alpha,\beta(\varepsilon_n)} - \overline{Y}_T^{x,\alpha}|^2 \underset{n\to+\infty}{\to} 0.$$

First we write:

$$\begin{aligned}|\overline{v}^{\alpha,\beta(\varepsilon_n)}(V_s^{x,\beta(\varepsilon_n)}) - \overline{v}^\alpha(V_s^x)| &\leq |\overline{v}^{\alpha,\beta(\varepsilon_n)}(V_s^{x,\beta(\varepsilon_n)}) - \overline{v}^{\alpha,\beta(\varepsilon_n)}(V_s^x)| \\ &\quad + |\overline{v}^{\alpha,\beta(\varepsilon_n)}(V_s^x) - \overline{v}^\alpha(V_s^x)| \\ &\leq C(1+|V_s^{x,\beta(\varepsilon_n)}|^2 + |V_s^x|^2)|V_s^{x,\beta(\varepsilon_n)} - V_s^x| \\ &\quad + |\overline{v}^{\alpha,\beta(\varepsilon_n)}(V_s^x) - \overline{v}^\alpha(V_s^x)|,\end{aligned}$$

which shows the convergence of $\overline{v}^{\alpha,\beta(\varepsilon_n)}(V_s^{x,\beta(\varepsilon_n)})$ toward $\overline{v}^\alpha(V_s^x)$ almost surely, up to a subsequence (it is well known that $\forall T>0$, $E\sup_{0\leq t\leq T}|V_t^{x,\beta(\varepsilon_n)} - V_t^x|^2 \underset{n\to+\infty}{\longrightarrow} 0$). Then, due to the fact that
$|\overline{v}^{\alpha,\beta(\varepsilon_n)}(V_s^{x,\beta(\varepsilon_n)})| \leq M_\psi/\alpha$ $\mathbb{P}$-a.s., we can apply the dominated convergence theorem to show that:

$$\mathbb{E}\int_0^T |\overline{Y}_s^{x,\alpha,\beta(\varepsilon_n)} - \overline{Y}_s^{x,\alpha}|^2 \mathrm{d}s \underset{n\to+\infty}{\to} 0 \text{ and } \mathbb{E}|\overline{Y}_T^{x,\alpha,\beta(\varepsilon_n)} - \overline{Y}_T^{x,\alpha}|^2 \underset{n\to+\infty}{\to} 0.$$

Now we show that $(Z^{x,\alpha,\beta(\varepsilon_n)})_n$ is Cauchy in $\mathscr{M}^2(\mathbb{R}_+,\mathbb{R}^{1\times d})$. We denote

$$\widetilde{V}_t = V_t^{x,\beta(\varepsilon_n)} - V_t^{x,\beta(\varepsilon_n)'};$$

$$\widetilde{Y}_t = \overline{Y}_t^{x,\alpha,\beta(\varepsilon_n)} - \overline{Y}_t^{x,\alpha,\beta(\varepsilon_n)'};$$

$$\widetilde{Z}_t = \overline{Z}_t^{x,\alpha,\beta(\varepsilon_n)} - \overline{Z}_t^{x,\alpha,\beta(\varepsilon_n)'};$$

and

$$\widetilde{\lambda} = \alpha v^{\alpha,\beta(\varepsilon_n)}(0) - \alpha v^{\alpha,\beta(\varepsilon_n)'}(0).$$

Itô's formula applied to $|\widetilde{Y}_t|^2$ gives us, for all $\varepsilon_1,\varepsilon_2,\varepsilon_3>0$:

$$\begin{aligned}|\widetilde{Y}_t|^2 + \int_t^T |\widetilde{Z}_s|^2\mathrm{d}s &= |\widetilde{Y}_T|^2 + 2\int_t^T \widetilde{Y}_t[\psi(V_s^{x,\beta(\varepsilon_n)}, Z_s^{x,\alpha,\beta(\varepsilon_n)}) - \psi(V_s^{x,\beta(\varepsilon_n)'}, Z_s^{x,\alpha,\beta(\varepsilon_n)'}) \\ &\quad - (\alpha\overline{Y}_s^{x,\alpha,\beta(\varepsilon_n)} - \alpha\overline{Y}_s^{x,\alpha,\beta(\varepsilon_n)'}) - \widetilde{\lambda}]\mathrm{d}s \\ &\quad - 2\int_t^T \widetilde{Y}_s\widetilde{Z}_s\mathrm{d}W_s \\ &\leq |\widetilde{Y}_T|^2 + (\varepsilon_1 M_\psi + \varepsilon_2 M_\psi + \varepsilon_3)\int_t^T |\widetilde{Y}_s|^2\mathrm{d}s + \frac{M_\psi}{\varepsilon_1}\int_t^T |\widetilde{V}_s|^2\mathrm{d}s \\ &\quad + \frac{M_\psi}{\varepsilon_2}\int_t^T |\widetilde{Z}_s|^2\mathrm{d}s + \frac{1}{\varepsilon_3}\int_t^T |\widetilde{\lambda}|^2\mathrm{d}s + c\int_t^T |\widetilde{Y}_s|\mathrm{d}s - 2\int_t^T \widetilde{Y}_s\widetilde{Z}_s\mathrm{d}W_s,\end{aligned}$$

because $\alpha|v^{\alpha,\varepsilon}(0)| \leq M_\psi$. Thus, taking the expectation and for ε_2 large enough we get

$$\mathbb{E}\int_0^T |\widetilde{Z}_s|^2\mathrm{d}s \leq \mathbb{E}|\widetilde{Y}_T|^2 + c\left(\mathbb{E}\left[\int_0^T |\widetilde{Y}_s|^2\mathrm{d}s\right] + \mathbb{E}\left[\int_0^T |\widetilde{V}_s|^2\mathrm{d}s\right] + \mathbb{E}\left[\int_0^T |\widetilde{Y}_s|\mathrm{d}s\right] + T|\widetilde{\lambda}|^2\right),$$

which proves that $(Z^{x,\alpha,\beta(\varepsilon_n)})_{\beta(\varepsilon_n)}$ is Cauchy in $\mathscr{M}^2(\mathbb{R}_+,\mathbb{R}^{1\times d})$. Now we pass to the limit in equation (2.20) to obtain:

$$Y_t^{x,\alpha} = Y_T^{x,\alpha} + \int_t^T (\psi(V_s^x, Z_s^{x,\alpha}) - \overline{\lambda}^\alpha)\mathrm{d}s - \int_t^T Z_s^{x,\alpha}\mathrm{d}W_s, \quad 0 \le t \le T < +\infty.$$

Now we reiterate the above method. Thanks to the following estimates: $\forall x, x' \in \mathbb{R}^d$,

$$|\overline{v}^\alpha(x) - \overline{v}^\alpha(x')| \le C(1 + |x|^2 + |x'|^2)|x - x'|,$$

and

$$|\overline{\lambda}^\alpha| \le M_\psi,$$

it is possible, by a diagonal procedure, to construct a sequence $(\alpha_n)_n$ such that

$$\overline{v}^{\alpha_n}(x) \underset{n\to+\infty}{\longrightarrow} \overline{v}(x)$$
$$\overline{\lambda}^{\alpha_n} \underset{n\to+\infty}{\longrightarrow} \overline{\lambda}.$$

We define $\overline{Y}_t^x := \overline{v}(V_t^x)$. Let us just precise why

$$\mathbb{E}\int_0^T |\overline{Y}_s^{x,\alpha} - \overline{Y}_s^x|^2\mathrm{d}s \underset{n\to+\infty}{\to} 0 \text{ and } \mathbb{E}|\overline{Y}_T^{x,\alpha} - \overline{Y}_T^x|^2 \underset{n\to+\infty}{\to} 0.$$

First the convergence of $\overline{v}^{\alpha_n}(V_s^x)$ toward $\overline{v}(V_s^x)$ is clear. Secondly, we have

$$|v^{\alpha_n}(V_s^x)| \le C(1 + |V_s^x|^2).$$

Therefore the dominated convergence theorem can be applied to show that:

$$\mathbb{E}\int_0^T |\overline{Y}_s^{x,\alpha,\beta(\varepsilon_n)} - \overline{Y}_s^{x,\alpha}|^2\mathrm{d}s \underset{n\to+\infty}{\to} 0 \text{ and } \mathbb{E}|\overline{Y}_T^{x,\alpha,\beta(\varepsilon_n)} - \overline{Y}_T^{x,\alpha}|^2 \underset{n\to+\infty}{\to} 0.$$

Then, just as before, it is possible to show that $(Z^{x,\alpha_n})_{\alpha_n}$ is Cauchy in $\mathscr{M}^2(\mathbb{R}_+,\mathbb{R}^{1\times d})$. We denote its limit by $\overline{Z}_s^x$.

The end of the proof is very classical, it suffices to apply BDG's inequality to show that $\mathbb{E}\sup_{0\le t\le T}|\overline{Y}^x|^2 < +\infty$, $\forall T > 0$. To show that $\overline{Z}^x$ is Markovian, just apply the same method as in the proof of Theorem 4.4 in [31]. □

Remark 2.12. It is clear that we do not have uniqueness of the solutions of EBSDE (2.9) because if (Y, Z, λ) is a solution then $(Y + \theta, Z, \lambda)$ is another solution, for all $\theta \in \mathbb{R}$. However we have a uniqueness property for λ under the following polynomial growth property:

$$|Y_t^x| \le C(1 + |V_t^x|^2).$$

One can notice that the solution $\overline{Y}_t^x = \overline{v}(V_t^x)$ constructed in the proof of Theorem (2.11) satisfies such a growth property.

Theorem 2.13. *(Uniqueness of λ). Assume that Hypotheses (2.1) and (2.2) hold true. Let us suppose that we have two solutions of EBSDE (2.23) denoted by (Y, Z, λ) and (Y', Z', λ') where Y and Y' are progressively measurable continuous processes, Z and Z' $\in \mathscr{M}^2(\mathbb{R}_+,\mathbb{R}^{1\times d})$ and λ, $\lambda' \in \mathbb{R}$. Finally assume that the following growth properties hold:*

$$|Y_t| \le C(1 + |V_t^x|^2)$$
$$|Y'_t| \le C'(1 + |V_t^x|^2).$$

Then $\lambda = \lambda'$.

Proof. It suffices to adapt the proof of Theorem 4.6 of [31]. With the same notations one can write:

$$\begin{aligned}\widetilde{\lambda} &= T^{-1}\mathbb{E}^{\mathbb{P}_h}[\widetilde{Y}_T - \widetilde{Y}_0] \\ &\leq T^{-1}\mathbb{E}^{\mathbb{P}_h}((C+C')(1+|V_T^x|^2)) + T^{-1}\mathbb{E}^{\mathbb{P}_h}((C+C')(1+|x|^2)).\end{aligned}$$

To conclude, just use the estimates from Lemma 2.1, and let $T \to +\infty$. □

2.4 The ergodic BSDE with zero and non-zero Neumann boundary conditions in a weakly dissipative environment

In this section we replace the process $(V_t^x)_{t\geq 0}$ by the process $(X_t^x)_{t\geq 0}$, which is solution of a stochastic differential equation reflected in the closure of an open convex subset G of $\mathbb{R}^d$ with regular boundary, namely, we consider the following stochastic equation for a pair of unknown processes $(X_t^x, K_t^x)_{t\geq 0}$ such that, for every $x \in \overline{G}$, $t \geq 0$:

$$\begin{cases} X_t^x = x + \displaystyle\int_0^t f(X_s^x)\mathrm{d}s + \int_0^t \sigma(X_s^x)\mathrm{d}W_s + \int_0^t \nabla\phi(X_s^x)\mathrm{d}K_s^x, \\ K_t^x = \displaystyle\int_0^t \mathbb{1}_{\{X_s^x\in\partial G\}}\mathrm{d}K_s^x, \end{cases} \tag{2.21}$$

where f is weakly dissipative.

As far as we know, there is no result regarding such diffusions. That is why it is necessary to adapt a result of Menaldi in [60] where an existence and uniqueness result is stated by a penalization method for a diffusion reflected in a convex and bounded set under Lipschitz assumptions for the drift. Therefore, it is necessary to adapt this result in our framework, namely when the set is not bounded anymore but with weakly dissipative assumptions for the drift.

We denote by $\Pi(x)$ the projection of $x \in \mathbb{R}^d$ on $\overline{G}$. Let us denote by $(X_t^{x,n})_{t\geq 0}$ the unique strong solution of the following penalized problem associated to the reflected problem (2.21) :

$$X_t^{x,n} = x + \int_0^t (d + F_n + b)(X_s^{x,n})\mathrm{d}s + \int_0^t \sigma(X_s^{x,n})\mathrm{d}W_s, \tag{2.22}$$

where $\forall x \in \mathbb{R}^d$, $\quad F_n(x) = -2n(x - \Pi(x))$.

Remark 2.14. The functions $d+F_n+b$ and σ satisfy Hypothesis 2.1. Indeed, from [37], F_n is 0-dissipative therefore $d+F_n$ remains η-dissipative thus the estimate of Lemma 2.5 holds with constants which do not depend on n. Furthermore one can remark that for all $\xi \in \mathbb{R}^d$, ${}^t\xi\nabla F_n(x)\xi \leq 0$, for all $x \in \mathbb{R}^d$ (see for example [37]). Finally, taking $a \in \overline{G}$ (thus $F_n(a) = 0$) in Remark 2.3 shows us that the estimate of Lemma 2.1 holds with constants that do not depend on n.

We need the following assumptions on G:

Hypothesis 2.4. G is an open convex set of $\mathbb{R}^d$.

Hypothesis 2.5. There exists a function $\phi \in \mathscr{C}_b^2(\mathbb{R}^d)$ such that $G = \{\phi > 0\}$, $\partial G = \{\phi = 0\}$ and $|\nabla\phi(x)| = 1$, $\forall x \in \partial G$.

The following Lemma states that the penalized process is Cauchy in the space of predictable continuous process for the norm $\mathbb{E}\sup_{0\leq t\leq T}|\cdot|^p$, for every $p>2$ and that it converges to the reflected process solution of (2.21) for a process K^x with bounded variations.

Lemma 2.15. *Assume that Hypotheses (2.1), (2.4) and (2.5) hold true. Then for every $x\in\overline{G}$, there exists a unique pair of processes $\{(X_t^x,K_t^x)_{t\geq 0}\}$ with values in $(\overline{G}\times\mathbb{R}_+)$ and which belong to the space $\mathscr{S}^p\times\mathscr{S}^p$, $\forall 1\leq p<+\infty$, satisfying (2.21) and such that*

$$\eta_t^x := \int_0^t \nabla\phi(X_s^x)\mathrm{d}K_s^x \qquad \textit{has bounded variation on } [0,T],\ 0<T<\infty,\ \eta_0^x=0$$

and for all process z continuous and progressively measurable taking values in the closure $\overline{G}$ we have

$$\int_0^T (X_s^x - z_s)\mathrm{d}\eta_s^x \leq 0, \qquad \forall T>0.$$

Finally the following estimate hold for the convergence of the penalized process, for any $1<q<p/2$, for any $T\geq 0$ there exists $C>0$ such that

$$\mathbb{E}\sup_{0\leq t\leq T}|X_t^{x,n}-X_t^x|^p \leq C\left(\frac{1}{n^q}\right),$$

Proof. The proof is given in Appendix. □

2.4.1 The ergodic BSDE with zero Neumann boundary conditions in a weakly dissipative environment

In a first time we are concerned with the following EBSDE with zero Neumann condition in infinite horizon:

$$Y_t^x = Y_T^x + \int_t^T [\psi(X_s^x,Z_s^x)-\lambda]\mathrm{d}s - \int_t^T Z_s^x\mathrm{d}W_s,\ \forall 0\leq t\leq T<+\infty, \tag{2.23}$$

where the unknown is the triplet $(Y_\cdot^x,Z_\cdot^x,\lambda)$. $(X_t^x)_{t\geq 0}$ is the solution of (2.21).

To study the problem of existence of a solution to such an equation, we are going to study the following BSDE, with monotonic drift in y: $\forall 0\leq t\leq T<+\infty$,

$$Y_t^{x,\alpha,n,\varepsilon} = Y_T^{x,\alpha,n,\varepsilon} + \int_t^T [\psi(X_s^{x,n,\varepsilon},Z_s^{x,\alpha,n,\varepsilon})-\alpha Y_s^{x,\alpha,n,\varepsilon}]\mathrm{d}s - \int_t^T Z_s^{x,\alpha,n,\varepsilon}\mathrm{d}W_s, \tag{2.24}$$

where the process $(X_t^{x,n,\varepsilon})$ is the solution of the following SDE:

$$X_t^{x,n,\varepsilon} = x + \int_0^t (f^\varepsilon(X_s^{x,n,\varepsilon}) + F_n^\varepsilon(X_s^{x,n,\varepsilon}))\mathrm{d}s + \int_0^t \sigma(X_s^{x,n})\mathrm{d}W_s.$$

Remark 2.16. F_n is regularized like other regularized functions. Thanks to convolution arguments it is possible to construct a sequence of functions F_n^ε which converges pointwisely toward F_n and such that for all ε, F_n^ε is 0-dissipative and $4n$-Lipschitz.

Now we can state the existence theorem for EBSDE (2.23).

Theorem 2.17. *Assume that Hypotheses 2.1, 2.2, 2.4 and 2.5 hold. Then there exists a solution $(\overline{Y}_t^x,\overline{Z}_t^x,\overline{\lambda})$ to EBSDE (2.23) such that $\overline{Y}_\cdot^x=\overline{v}(V_\cdot^x)$ with $\overline{v}$ locally Lipschitz, and there exists a measurable function $\overline{\xi}:\mathbb{R}^d\to\mathbb{R}^{1\times d}$ such that $\overline{Z}^x\in\mathscr{M}^2(\mathbb{R}_+,\mathbb{R}^{1\times d})$ and $\overline{Z}_\cdot^x=\overline{\xi}(X_\cdot^x)$.*

Proof. We give the main ideas, because the proof is very similar to the proof of Theorem 2.11. The beginning of the proof is the same as the proof of Theorem 2.11. Lemma 2.8 gives us the existence and uniqueness of the solution $(Y^{x,\alpha,n,\varepsilon}, Z^{x,\alpha,n,\varepsilon})$ of BSDE (2.24) in $\mathscr{S}^2 \times \mathscr{M}^2(\mathbb{R}_+, \mathbb{R}^{1\times d})$. Then, as the function $d + F_n$ is still η-dissipative and as the work in the previous section involves d only through its dissipativity constant η, we can apply previous results. As always we define $v^{\alpha,n,\varepsilon}(x) := Y_0^{\alpha,n,\varepsilon}$. By Lemma 2.10 we have the following estimate: $\forall x, x' \in \mathbb{R}^d$:

$$|v^{\alpha,n,\varepsilon}(x) - v^{\alpha,n,\varepsilon}(x')| \leq C(1 + |x|^2 + |x'|^2)|x - x'|.$$

In addition we also have

$$|\alpha v^{\alpha,n,\varepsilon}(0)| \leq M_\psi.$$

As those inequalities are uniform in ε it is possible to construct by a diagonal procedure a subsequence $\varepsilon_p \to +0$ such that $\forall n \in \mathbb{N}, \alpha > 0$:

$$\overline{v}^{\alpha,n,\varepsilon_p}(x) \underset{p\to+\infty}{\longrightarrow} \overline{v}^{\alpha,n}(x) \quad \text{and} \quad \alpha v^{\alpha,n,\varepsilon_p(0)} \underset{p\to+\infty}{\longrightarrow} \overline{\lambda}^{\alpha,n},$$

We recall the fact that the function $\overline{v}^{\alpha,n}$ is locally Lipschitz on $\mathbb{R}^d$ and that we keep the following estimates:

$$|\overline{v}^{\alpha,n}(x) - \overline{v}^{\alpha,n}(x')| \leq C(1 + |x|^2 + |x'|^2)|x - x'|;$$

$$|\overline{\lambda}^{\alpha,n}| \leq M_\psi.$$

Now let us define $\forall t \geq 0$, $\overline{Y}_t^{x,\alpha,n} := \overline{v}^{\alpha,n}(V_t^{x,n})$. Let us show that

$$\mathbb{E}\int_0^T |\overline{Y}_s^{x,\alpha,n,\varepsilon_p} - \overline{Y}_s^{x,\alpha,n}|^2 \mathrm{d}s \underset{p\to+\infty}{\to} 0 \text{ and } \mathbb{E}|\overline{Y}_T^{x,\alpha,n,\varepsilon_p} - \overline{Y}_T^{x,\alpha,n}|^2 \underset{p\to+\infty}{\to} 0.$$

First we write:

$$\begin{aligned}|\overline{v}^{\alpha,n,\varepsilon_p}(X_s^{x,n,\varepsilon_p}) - \overline{v}^{\alpha,n}(X_s^{x,n})| &\leq |\overline{v}^{\alpha,n,\varepsilon_p}(X_s^{x,n,\varepsilon_p}) - \overline{v}^{\alpha,n,\varepsilon_p}(X_s^{x,n})| \\ &\quad + |\overline{v}^{\alpha,n,\varepsilon_p}(X_s^{x,n}) - \overline{v}^{\alpha,n}(X_s^{x,n})| \\ &\leq C(1 + |X_s^{x,n,\varepsilon_p}|^2 + |X_s^{x,n}|^2)|X_s^{x,n,\varepsilon_p} - X_s^{x,n}|,\end{aligned}$$

which shows the pointwise convergence of $\overline{v}^{\alpha,n,\varepsilon_p}(V_s^{x,n,\varepsilon_p})$ toward $\overline{v}^{\alpha,n}(V_s^{x,n})$ almost surely when $p \to +\infty$. Then, due to the fact that $|\overline{v}^{\alpha,\beta(\varepsilon_n)}(V_s^{x,\beta(\varepsilon_n)})| \leq M_\psi/\alpha$ $\mathbb{P}$-a.s., we can apply the dominated convergence theorem to show that:

$$\mathbb{E}\int_0^T |\overline{Y}_s^{x,\alpha,n,\varepsilon_p} - \overline{Y}_s^{x,\alpha,n}|^2 \mathrm{d}s \underset{p\to+\infty}{\to} 0 \text{ and } \mathbb{E}|\overline{Y}_T^{x,\alpha,n,\varepsilon_p} - \overline{Y}_T^{x,\alpha,n}|^2 \underset{p\to+\infty}{\to} 0.$$

In addition it is possible to show as in Theorem 2.11 that $(Z^{x,\alpha,n,\varepsilon_p})_p$ is Cauchy in $\mathscr{M}^2(\mathbb{R}_+, \mathbb{R}^{1\times d})$.

Note that we keep the estimates $\forall x, x' \in \mathbb{R}^d$:

$$|\overline{v}^{\alpha,n}(x) - \overline{v}^{\alpha,n}(x')| \leq C(1 + |x|^2 + |x'|^2)|x - x'|,$$

and

$$|\overline{\lambda}^{\alpha,n}| \leq M_\psi.$$

Therefore, again, by a diagonal procedure, it is possible to extract a subsequence $(\beta(n))_n$ such that

$$v^{\alpha,\beta(n)}(x) \to \overline{v}^{\alpha}(x).$$

And thanks to Lemma 2.15, one can apply the dominated convergence theorem to show that:

$$\mathbb{E}\int_0^T |\overline{Y}_s^{x,\alpha,\beta(n)} - \overline{Y}_s^{x,\alpha}|^2 \mathrm{d}s \underset{n\to+\infty}{\to} 0 \text{ and } \mathbb{E}|\overline{Y}_T^{x,\alpha,\beta(n)} - \overline{Y}_T^{x,\alpha}|^2 \underset{n\to+\infty}{\to} 0.$$

Finally a last diagonal procedure in α allows us to conclude (see the end of the proof of Theorem 2.11).

□

Once again, we notice that the solution we have constructed satisfies the following growth property:

$$|\overline{Y}_t^x| \le C(1+|X_t^x|^2),$$

so it is natural to establish the following theorem under the same growth properties.

Theorem 2.18. *(Uniqueness of λ). Assume that Hypotheses 2.1 and 2.2 hold true. Let (Y, Z, λ) be a solution of EBSDE (2.23). Then λ is unique among solutions (Y, Z, λ) such that Y is a bounded continuous process and $Z \in \mathscr{M}^2(\mathbb{R}_+, \mathbb{R}^{1\times d})$. Finally assume that we have the following growth property*

$$|Y_t| \le C(1+|X_t^x|^2),$$
$$|Y'_t| \le C'(1+|X_t^x|^2).$$

Then $\lambda = \lambda'$.

Proof. Simply, adapt the proof of Theorem 4.6 of [31]. With the same notations we can write:

$$\begin{aligned}\widetilde{\lambda} &= T^{-1}\mathbb{E}^{\mathbb{P}_h}[\widetilde{Y}_T - \widetilde{Y}_0]\\ &\le (C+C')T^{-1}(2+|x|^2+\mathbb{E}^{\mathbb{P}_h}|X_T^x|^2)\\ &\le (C+C')T^{-1}(2+|x|^2+\mathbb{E}^{\mathbb{P}_h}|X_T^{x,n}|^2+\mathbb{E}^{\mathbb{P}_h}|X_T^x - X_T^{x,n}|^2)\end{aligned}$$

To conclude, just use the first estimate from Lemma 2.1, the estimate from Lemma 2.15 and let $T \to +\infty$. □

2.4.2 The ergodic BSDE with non-zero Neumann boundary conditions in a weakly dissipative environment

We are now concerned by the following EBSDE in infinite horizon:

$$Y_t^x = Y_T^x + \int_t^T [\psi(X_s^x, Z_s^x) - \lambda]\mathrm{d}s + \int_t^T [g(X_s^x) - \mu]\mathrm{d}K_s^x - \int_t^T Z_s^x \mathrm{d}W_s,\ \forall 0 \le t \le T < +\infty, \tag{2.25}$$

where $g : \mathbb{R}^d \to \mathbb{R}$ is measurable and such that the term $\int_t^T [g(X_s^x) - \mu]\mathrm{d}K_s^x$ is well defined for all $0 \le t \le T < +\infty$.

Proposition 2.19. *(Existence of a Solution (Y, Z, λ)). Assume that the hypothesis 2.1, 2.2 and 2.5 hold true. Then for any $\mu \in \mathbb{R}$ there exists $\lambda \in \mathbb{R}$, Y^x continuous adapted process and $Z^x \in \mathscr{M}^2(\mathbb{R}_+, \mathbb{R}^{1\times d})$ such that the triple (Y, Z, λ) is a solution of EBSDE (2.25).*

Proof. The Theorem 2.17 gives us the existence of a solution (Y^x, Z^x, λ) of the following EBSDE

$$Y_t^x = Y_T^x + \int_t^T [\psi(X_s^x, Z_s^x) - \lambda]\mathrm{d}s - \int_t^T Z_s^x \mathrm{d}W_s, \qquad \forall 0 \le t \le T < +\infty. \tag{2.26}$$

Now, defining $\widehat{Y}_t^x = Y_t^x - \int_0^t [g(X_s^x) - \mu]\mathrm{d}K_s^x$, it is easy to see that $(\widehat{Y}^x, Z^x, \lambda)$ is a solution of the EBSDE (2.25) with μ fixed. □

Remark 2.20. The constructed solution $\widehat{Y}^x$ is not Markovian anymore. Furthermore, it satisfies the following growth property: $\forall t \ge 0$, $|\widehat{Y}_t^x| \le C(1 + |X_t^x|^2 + K_t^x)$. This dependence on K_t^x prevents us to get the uniqueness of λ among the space of solutions satisfying such a growth property.

Similarly, for every $\lambda \in \mathbb{R}$, an existence result can be stated for a solution (Y, Z, μ).

Proposition 2.21. *(Existence of a Solution (Y, Z, μ)). Assume that Hypotheses 2.1, 2.2 and 2.5 hold true. Then for any $\lambda \in \mathbb{R}$ there exist a continuous adapted process Y and $Z^x \in \mathscr{M}^2(\mathbb{R}_+, \mathbb{R}^{1\times d})$ such that for all $\mu \in \mathbb{R}$ the triple (Y, Z, μ) is a solution of EBSDE (2.25).*

Proof. From Theorem 2.17, we have constructed a solution $(Y^{x,0}, Z^{x,0}, \lambda^0)$ of the following EBSDE

$$Y_t^{x,0} = Y_T^{x,0} + \int_t^T [\psi(X_s^x, Z_s^{x,0}) - \lambda^0]\mathrm{d}s - \int_t^T Z_s^{x,0} \mathrm{d}W_s, \qquad \forall 0 \le t \le T < +\infty. \tag{2.27}$$

Then setting $\widehat{Y}_t^x := Y_t^{x,0} + (\lambda - \lambda^0)t - \int_0^t [g(X_s^x) - \mu]\mathrm{d}K_s^x$, the triple $(\widehat{Y}^x, Z^{x,0}, \mu)$ is solution of EBSDE (2.25). □

Remark 2.22. The constructed solution satisfies the following growth property:

$$|\widehat{Y}_t| \le C(1 + |X_t^x|^2 + K_t^x + t), \mathbb{P}\text{-a.s.}$$

Again, this solution does not allow us to establish a result of uniqueness for μ among the space of solutions satisfying such a growth property.

Remark 2.23. If the convex $\overline{G}$ is assumed to be bounded, it is possible, following [72] to show that there exists a Markovian solution (Y, Z, λ) when μ is fixed or (Y, Z, μ) when λ is fixed exists, for a driver weakly dissipative. The proofs are the same as in [72].

2.4.3 Probabilistic interpretation of the solution of an elliptic PDE with zero Neumann boundary condition

We are concerned with the following semi-linear elliptic PDE:

$$\begin{cases} \mathscr{L}v(x) + \psi(x, \nabla v(x)\sigma(x)) = \lambda, & x \in G, \\ \frac{\partial v}{\partial n}(x) = 0, & x \in \partial G, \end{cases} \tag{2.28}$$

where:

$$\mathscr{L}u(x) = \frac{1}{2}\operatorname{Tr}(\sigma(x)^t\sigma(x)\nabla^2 u(x)) + \nabla u(x) f(x).$$

The unknowns of this equation is the couple (v, λ). Now we show that the pair (v, λ) defined in Theorem 2.19 is a viscosity solution of the PDE (2.28).

Theorem 2.24. *Assume that hypotheses of Theorem 2.17 hold . Then (v, λ) is a viscosity solution of the elliptic PDE (2.28) where (v, λ) is defined in Theorem 2.17.*

Proof. Just adapt the proof of Theorem 4.3 from [68]. □

2.4.4 Optimal ergodic control

We make the standard assumption for optimal ergodic control, namely we consider U a separable metric space, which is the state space of the control process ρ. ρ is assumed to be $(\mathscr{F}_t)$-progressively measurable. We introduce $R : U \to \mathbb{R}^d$ and $L : \mathbb{R}^d \times U \to \mathbb{R}$ two continuous functions such that , for some constants $M_R > 0$ and $M_L > 0$, $\forall u \in U, \forall x, x' \in \mathbb{R}^d$,

- $|R(u)| \leq M_R$,
- $|L(x, u)| \leq M_L$,
- $|L(x, u) - L(x', u)| \leq M_L|x - x'|$.

For an arbitrary control ρ, the cost will be evaluated relatively to the following Girsanov density:

$$\Gamma_T^\rho = \exp\left(\int_0^T R(\rho_s)\mathrm{d}W_s - \frac{1}{2}\int_0^T |R(\rho_s)|^2\mathrm{d}s\right).$$

We denote by $\mathbb{P}_T^\rho$ the associated probability measure, namely: $\mathrm{d}\mathbb{P}_T^\rho = \Gamma_T^\rho \mathrm{d}\mathbb{P}$ on $\mathscr{F}_T$. Now we define the ergodic costs, relatively to a given control ρ and a starting point $x \in \mathbb{R}^d$, by:

$$I(x, \rho) = \limsup_{T\to+\infty} \frac{1}{T}\mathbb{E}_T^\rho\left[\int_0^T L(X_s^x, \rho_s)\mathrm{d}s\right], \tag{2.29}$$

where $\mathbb{E}_T^\rho$ denotes expectation with respect to $\mathbb{P}_T^\rho$. We notice that the process $W_t^\rho := W_t - \int_0^t R(\rho_s)\mathrm{d}s$ is a Wiener process on $[0, T]$ under $\mathbb{P}_T^\rho$. We define the Hamiltonian in the usual way:

$$\psi(x, z) = \inf_{u\in U}\{L(x, u) + zR(u)\}, \qquad x \in R^d, \quad z \in \mathbb{R}^{1\times d}, \tag{2.30}$$

and we remark that if, for all x, z, the infimum is attained in (2.30) then, according to Theorem 4 of [59], there exists a measurable function $\gamma : \mathbb{R}^d \times \mathbb{R}^{1\times d} \to U$ such that:

$$\psi(x, z) = L(x, \gamma(x, z)) + zR(\gamma(x, z)). \tag{2.31}$$

One can verify that γ is a Lipchitz function. Now we can prove the following theorem, exactly like in [72].

Theorem 2.25. *Assume that the hypotheses of Theorem 2.17 hold true. Let (Y, Z, λ) be a solution of EBSDE (2.25) with μ fixed. Then:*

1. *For arbitrary control ρ we have $I(x, \rho) \geq \lambda$.*
2. *If $L(X_t^x, \rho_s) + Z_t^x R(\rho_t) = \psi(X_t^x, Z_t^x)$, $\mathbb{P}$-a.s. for almost every t then $I(x, \rho) = \lambda$.*
3. *If the infimum is attained in (2.31) then the control $\overline{\rho}_t = \gamma(X_t^x, Z_t^x)$ verifies $I(x, \overline{\rho}) = \lambda$.*

Remark 2.26. When the Neumann conditions are different from 0, we need regularity on the solution Y_t^x in order to state the same result. Again the degeneracy of the solution constructed in Proposition 2.19 or 2.21 does not allow us to conclude.

2.5 Appendix

2.5.1 Proof of Lemma 2.1

Let us define $\varphi(x) = |x-a|^p$ for $p \geq 1$. We recall the following formulas for derivatives of φ, for $p \geq 2$.

$$\nabla\varphi(x) = p(x-a)|x-a|^{p-2}.$$

$$\partial^2\varphi(x)/\partial x_i \partial x_j = \begin{cases} p|x-a|^{p-2} + p(p-2)(x_i-a_i)^2|x-a|^{p-4} & \text{if } i = j, \\ p(p-2)(x_i-a_i)(x_j-a_j)|x-a|^{p-4} & \text{if } i \neq j. \end{cases}$$

Therefore we have the following estimate

$$|\nabla^2\varphi(x)| \leq K|x-a|^{p-2}, \tag{2.32}$$

for a constant K which depends only on p and d. Under the hypothesis of this Lemma, it is well known that a unique strong solution for which the explosion time is almost surely equal to infinity exists (see [57] for example). By Itô's formula we get, for $p = 2$, for all $t \geq 0$,

$$\begin{aligned} |V_t^x - a|^2 e^{2\eta_1 t} &= |x-a|^2 + 2\int_0^t e^{2\eta_1 s}(V_s^x - a, f(V_s^x)\mathrm{d}s + \sigma(V_s^x)\mathrm{d}W_s) \\ &\quad + 2\eta_1 \int_0^t |V_s^x - a|^2 e^{2\eta_1 s}\mathrm{d}s + \int_0^t \sum_i (\sigma(V_s^x)^t\sigma(V_s^x))_{i,i} e^{2\eta_1 s}\mathrm{d}s \\ &\leq |x-a|^2 + 2\int_0^t {}^t(V_s^x - a)\sigma(V_s^x)\mathrm{d}W_s + \frac{2\eta_2 + d|\sigma|_\infty}{2\eta_1}(e^{2\eta_1 t} - 1). \end{aligned} \tag{2.33}$$

Taking the expectation, we get:

$$\mathbb{E}|V_t^x - a|^2 \leq |x-a|^2 e^{-2\eta_1 t} + \frac{2\eta_2 + d|\sigma|_\infty^2}{2\eta_1}(1 - e^{-2\eta_1 t}). \tag{2.34}$$

Therefore:

$$\mathbb{E}|V_t^x|^2 \leq C(1 + |x|^2 e^{-2\eta_1 t}), \tag{2.35}$$

where C is a constant that depends only on a, η_1, η_2 and σ but not on the time t.

Let $0 < \delta < p\eta_1$. For $p > 2$, Itô's formula gives us, for a generic constant C which depends only on p, d, $|\sigma|_\infty$, η_2, ε (defined later):

$$\begin{aligned} |V_t^x - a|^p e^{(p\eta_1 - \delta)t} &\leq |x-a|^p + p\int_0^t e^{(p\eta_1-\delta)s}|V_s^x - a|^{p-2}(V_s^x - a, f(V_s^x)\mathrm{d}s + \sigma(V_s^x)\mathrm{d}W_s) \\ &\quad + (p\eta_1 - \delta)\int_0^t |V_s^x - a|^p e^{(p\eta_1-\delta)}\mathrm{d}s \\ &\quad + \frac{1}{2}\int_0^t \mathrm{Tr}(\sigma(V_s^x)^t\sigma(V_s^x)\nabla^2\varphi(V_s^x))e^{(p\eta_1-\delta)s}\mathrm{d}s. \end{aligned}$$

Then, taking the expectation, using the assumption on f and using estimate (2.32) we have

$$\begin{aligned} \mathbb{E}|V_t^x - a|^p e^{(p\eta_1-\delta)t} &\leq |x-a|^p + C\int_0^t \mathbb{E}|V_s^x - a|^{p-2}e^{(p\eta_1-\delta)s}\mathrm{d}s \\ &\quad - \delta\int_0^t \mathbb{E}|V_s^x - a|^p e^{(p\eta_1-\delta)s}\mathrm{d}s. \end{aligned}$$

Young's inequality $ab \leq a^p/p + b^q/q$ for $1/p + 1/q = 1$ with p replaced by $p/(p-2)$ and q replaced by $p/2$ applied to the last term of the above inequality allows us to write:

$$|V_s^x - a|^{p-2} \leq (p-2)\varepsilon|V_s^x - a|^p/p + 2/(p\varepsilon^{(p-2)/2}),$$

hence,

$$\begin{aligned}\mathbb{E}|V_t^x - a|^p e^{(p\eta_1-\delta)t} \leq |x-a|^p + \varepsilon C \int_0^t \mathbb{E}|V_s^x - a|^p e^{(p\eta_1-\delta)s}\mathrm{d}s + C/\varepsilon^{(p-2)/2} \\ - \delta \int_0^t \mathbb{E}|V_s^x - a|^p e^{(p\eta_1-\delta)s}\mathrm{d}s.\end{aligned}$$

We choose $\varepsilon = \delta/C$, then:

$$\mathbb{E}|V_t^x - a|^p e^{(p\eta_1-\delta)t} \leq |x-a|^p C.$$

Therefore:

$$\mathbb{E}|V_t^x - a|^p \leq C(1 + |x|^p e^{-(p\eta_1-\delta)t}).$$

This can be rewritten:

$$\mathbb{E}|V_t^x|^p \leq C(1 + |x|^p e^{-(p\eta_1-\delta)t}),$$

where C is a constant which depends on p, d, σ_∞, η_1, η_2, ε and a. Finally, note that this result holds for any $0 < \delta < p\eta_1$.

Now, let us come back to (2.33), we have, for all $r > 1$, $2r = p$

$$\begin{aligned}\mathbb{E}[\sup_{0\leq t\leq T} |V_t^x|^p] &\leq C\left(1 + |x|^p + \mathbb{E}\sup_{0\leq t\leq T}\left|\int_0^t {}^t(V_s^x - a)\sigma(V_s^x)\mathrm{d}W_s\right|^p\right) \\ &\leq C\left[1 + |x|^p + \mathbb{E}\left(\left(\int_0^T |{}^t(V_s^x - a)\sigma(V_s^x)|^2\mathrm{d}s\right)^{r/2}\right)\right]\end{aligned}$$

by BDG's inequality. Now distinguish the cases $r/2 < 1$ or $r/2 \geq 1$, we readily get, for each $p > 2$,

$$\begin{aligned}\mathbb{E}[\sup_{0\leq t\leq T} |V_t^x|^p] &\leq C(T)(1 + |x|^p + |x|^r) \\ &\leq C(T)(1 + |x|^p).\end{aligned}$$

Once this is established, one can readily extend this estimate to the case $p \geq 1$. Indeed, for $0 < \alpha < 1$, we have, by Jensen's inequality

$$\begin{aligned}\mathbb{E}[\sup_{0\leq t\leq T} |V_t^x|^{p\alpha}] &\leq \left(\mathbb{E}\left[\sup_{0\leq t\leq T} |V_t^x|^p\right]\right)^\alpha \\ &\leq C(T)(1 + |x|^{p\alpha}).\end{aligned}$$

$\square$

2.5.2 Proof of Lemma 2.5

We adapt the proof of Theorem 2.4 from [26]. We give the full proof for reader convenience. In this proof, κ_i, $i = 0, 1...$ denotes a constant which depends only on η, B, σ_∞ and the dimension d. There are three steps in this proof. In the first step, we show that the process V^x enters a fixed ball quickly enough. In the second step, we construct a coupling of solutions starting from two different points in this ball and we show that the probability of the constructed solutions to be equal after a time T (given in the proof) is positive. Iterating this argument in step 3, we obtain the result.

Step 1 : By Remark 2.3, one can take $a = 0$ in equation (2.34) and then:

$$\mathbb{E}|V_t^x|^2 \leq |x|^2 e^{-\eta_1 t} + \kappa_1.$$

By the Markov property : $\forall k \in \mathbb{N}$,

$$\mathbb{E}[|V^x_{(k+1)T}|^2|\mathscr{F}_{kT}] \leq |V^x_{kT}|^2 e^{-\eta_1 T} + \kappa_1. \tag{2.36}$$

Let us define for $R \geq 0$,

$$C_k = \{|V^x_{kT}|^2 \geq R\}, \qquad B_k = \bigcap_{j=0}^{k} C_j.$$

By Markov's inequality

$$\begin{aligned}\mathbb{P}(C_k + 1|\mathscr{F}_{kT}) &\leq \frac{\mathbb{E}(|V^x_{(k+1)T}|^2|\mathscr{F}_{kT})}{R} \\ &\leq \frac{|V^x_{kT}|^2 e^{-\eta_1 T}}{R} + \frac{\kappa_1}{R}.\end{aligned} \tag{2.37}$$

Let us multiply (2.36) and (2.37) by $\mathbb{1}_{B_k}$ and let us take the expectation to obtain, since $\mathbb{1}_{B_{k+1}} \leq \mathbb{1}_{B_k}$

$$\begin{pmatrix} \mathbb{E}(|V^x_{(k+1)T}|^2 \mathbb{1}_{B_{k+1}}) \\ \mathbb{P}(B_{k+1}) \end{pmatrix} \leq A \begin{pmatrix} \mathbb{E}(|V^x_{kT}|^2 \mathbb{1}_{B_k}) \\ \mathbb{P}(B_k) \end{pmatrix}$$

where

$$A = \begin{pmatrix} e^{-\eta_1 T} & \kappa_1 \\ \frac{e^{-\eta_1 T}}{R} & \frac{\kappa_1}{R} \end{pmatrix}.$$

The eigenvalues of A are 0 and $e^{-\eta_1 T} + \frac{\kappa_1}{R}$. So for every $T > 0$ we can pick R large enough such that $\lambda_{T,R} := e^{-\eta_1 T} + \frac{\kappa_1}{R} < 1$.

We deduce that

$$\mathbb{P}(B_k) \leq \kappa_2 \left(\lambda_{T,R}\right)^k (1 + |x|^2).$$

Defining,

$$\tau = \inf\{kT; |V^x_{kT}|^2 \leq R\},$$

it follows that

$$\mathbb{P}(\tau \geq kT) \leq \mathbb{P}(B_k) \leq \kappa_2 (\lambda_{T,R})^k (1 + |x|^2).$$

Thus, if we pick $\alpha > 0$ small enough such that $e^{\alpha T}\lambda_{T,R} < 1$, we obtain:

$$\mathbb{E}(e^{\alpha\tau}) \leq \kappa_2(1+|x|^2)\sum_{k=0}^{+\infty}\left(e^{\alpha T}\lambda_{T,R}\right)^k \leq \kappa_3(1+|x|^2). \tag{2.38}$$

Step 2. In this step, we construct a coupling of processes starting respectively from $x, y \in B_R$, the ball of center 0 and radius R, such that the probability that they take the same value at time T is strictly positive. We denote by μ^x the law of V^x under $\mathbb{P}$ and by μ^y the law of V^y under $\mathbb{P}$ on $[0,T]$. Let us define

$$\widetilde{V}_t = V_t^y + Y_t^{x,y}$$

where $Y_t^{x,y}$ is the solution of the following SDE, for all $t < T$,

$$\begin{cases} \mathrm{d}Y_t^{x,y} = \left[d(V_t^y + Y_t^{x,y}) - d(V_t^y) - \sigma(V_t^y + Y_t^{x,y})\sigma^{-1}(V_t^y)\dfrac{LY_t^{x,y}}{T-t}\right]\mathrm{d}t \\ \qquad\qquad + \left[\sigma(V_t^y + Y_t^{x,y}) - \sigma(V_t^y)\right]\mathrm{d}W_t, \\ Y_0^{x,y} = x - y, \end{cases} \tag{2.39}$$

and where $L > 0$. We denote by $\widetilde{\mu}$ the law of $\widetilde{V}$ on $[0,T)$. The process $\widetilde{V}_t$ satisfies for all $t < T$:

$$\begin{cases} \mathrm{d}\widetilde{V}_t = f(\widetilde{V}_t)\mathrm{d}t + \sigma(\widetilde{V}_t)\left[\mathrm{d}W_t + \left(\sigma^{-1}(\widetilde{V}_t)\left(b(V_t^y) - b(\widetilde{V}_t)\right) - \sigma^{-1}(V_t^y)\dfrac{LY_t^{x,y}}{T-t}\right)\mathrm{d}t\right]. \\ \qquad\qquad + \left[\sigma(V_t^y + Y_t^{x,y}) - \sigma(V_t^y)\right]\mathrm{d}W_t, \\ \widetilde{V}_0 = x, \end{cases} \tag{2.40}$$

Since all the coefficients are locally Lipschitz, $Y_t^{x,y}$ (and thus $\widetilde{V}_t$) are well-defined continuous process for $t \leq T \wedge \zeta$ where ζ is the explosion time of $Y_t^{x,y}$; namely, $\zeta := \lim_{n\to+\infty}\zeta_n$ for

$$\zeta_n := \inf\{t \in [0,T) : |Y_t^{x,y}| \geq n\}$$

where we set $\inf\{\varnothing\} = T$. We define

$$h(t) = \sigma^{-1}(\widetilde{V}_t)\left(b(V_t^y) - b(\widetilde{V}_t)\right) - \sigma^{-1}(V_t^y)\frac{LY_t^{x,y}}{T-t}, \qquad t \leq T\wedge\zeta,$$

$$\mathrm{d}\widetilde{W}_t = \mathrm{d}W_t + h(t)\mathrm{d}t, \qquad t \leq T\wedge\zeta,$$

and

$$I_t = \int_0^t h(s)\mathrm{d}W_s, \qquad t \leq T\wedge\zeta.$$

If $\zeta = T$ and

$$R_t := \exp\left(-I_t - \frac{1}{2}\langle I, I\rangle_t\right)$$

is a uniformly integrable martingale for $t \in [0,T)$, then by the martingale convergence theorem, $R_T := \lim_{t\nearrow T} R_t$ exists and $(R_t)_{t\in[0,T]}$ is a martingale. In this case, by the

Girsanov theorem $(\widetilde{W}_t)_{t\in[0,T]}$ is a standard Brownian motion under the probability $R_T\mathbb{P}$. Rewrite (2.39) as

$$\begin{cases} \mathrm{d}Y_t^{x,y} = \Big[d(\widetilde{V}_t) - d(V_t^y) - \Big(\sigma(\widetilde{V}_t) - \sigma(V_t^y)\Big)\left(b(\widetilde{V}_t) - b(V_t^y)\right)\Big]\,\mathrm{d}t - \dfrac{LY_t^{x,y}}{T-t}\mathrm{d}t \\ \qquad\qquad + \Big[\sigma(\widetilde{V}_t) - \sigma(V_t^y)\Big]\,\mathrm{d}\widetilde{W}_t, \\ Y_0^{x,y} = x - y, \end{cases} \tag{2.41}$$

Now we would like to apply the Girsanov theorem. The following lemma is a direct adaptation of a result in [77]. However we give the proof for completeness.

Lemma 2.27. *Assume Hypothesis 2.1 hold true and let $x, y \in \mathbb{R}^d$ and $T > 0$ be fixed. Then*

1. *There holds*

$$\sup_{t\in[0,T),n\geq 1} \mathbb{E}R_{t\wedge\zeta_n}\log R_{t\wedge\zeta_n} < +\infty.$$

   Consequently,

$$R_{t\wedge\zeta} := \lim_{n\nearrow\infty} R_{t\wedge\zeta_n\wedge(T-1/n)}, \qquad t\in[0,T], \qquad R_{T\wedge\zeta} := \lim_{s\nearrow T} R_{s\wedge\zeta}$$

   exist such that $(R_{s\wedge\zeta})_{s\in[0,T]}$ is a uniformly martingale.
2. *Let $\mathbb{Q} = R_{T\wedge\zeta}\mathbb{P}$. Then $\mathbb{Q}(\zeta = T) = 1$ so that $\mathbb{Q} = R_T\mathbb{P}$.*
3. *$Y_T^{x,y} = 0$, $\quad\mathbb{Q}$-a.s.*

Proof. (1) let $t\in[0,T)$ and be fixed. By an Itô's formula,

$$\begin{aligned} \frac{|Y_{t\wedge\zeta_n}^{x,y}|^2}{T-s} &= \frac{|x-y|^2}{T} + 2\int_0^{t\wedge\zeta_n}\left(\frac{Y_s^{x,y}}{T-s}, d(\widetilde{V}_t) - d(V_s^y) - \frac{LY_s^{x,y}}{T-s}\right)\mathrm{d}s \\ &\quad - 2\int_0^{t\wedge\zeta_n}\left(\frac{Y_s^{x,y}}{T-s}, \Big(\sigma(\widetilde{V}_t) - \sigma(V_s^y)\Big)\left(b(\widetilde{V}_t) - b(V_t^y)\right)\right)\mathrm{d}s \\ &\quad + \int_0^{t\wedge\zeta_n}\frac{|Y_s^{x,y}|^2}{|T-s|^2}\mathrm{d}s + 2\int_0^{t\wedge\zeta_n}\left(\frac{Y_s^{x,y}}{T-s}, \Big[\sigma(\widetilde{V}_s) - \sigma(V_s^y)\Big]\,\mathrm{d}\widetilde{W}_s\right) \\ &\quad + \int_0^{t\wedge\zeta_n}\frac{1}{T-s}\,\mathrm{Tr}\,\Big[(\sigma(\widetilde{V}_s) - \sigma(V_s^y))^t(\sigma(\widetilde{V}_s) - \sigma(V_s^y))\Big]\,\mathrm{d}s \end{aligned}$$

By standard computations, since d is dissipative, since b is bounded and since σ is Lipschitz and bounded, for every $\varepsilon > 0$, we get

$$\begin{aligned} &\frac{|Y_{t\wedge\zeta_n}^{x,y}|^2}{T-s} + (2L-1-\varepsilon)\int_0^{t\wedge\zeta_n}\frac{|Y_s^{x,y}|^2}{|T-s|^2}\mathrm{d}s \\ &\qquad \leq \frac{|x-y|^2}{T} + \frac{C}{\varepsilon}(t\wedge\zeta_n) + 2\int_0^{t\wedge\zeta_n}\left(\frac{Y_s^{x,y}}{T-s}, \Big[\sigma(\widetilde{V}_s) - \sigma(V_s^y)\Big]\,\mathrm{d}\widetilde{W}_s\right). \end{aligned} \tag{2.42}$$

By the Girsanov theorem, $(\widetilde{W}_s)_{s\leq t\wedge\zeta_n}$ is a standard Brownian motion under the probability measure $R_{t\wedge\zeta_n}\mathbb{P}$. So, taking expectation $\mathbb{E}^{t,n}$ with respect to $R_{t\wedge\zeta_n}\mathbb{P}$, we arrive at

$$\mathbb{E}^{t,n}\int_0^{t\wedge\zeta_n}\frac{|Y_s^{x,y}|^2}{|T-s|^2}\mathrm{d}s \leq \frac{|x-y|^2}{T} + \frac{C}{\varepsilon}T, \qquad s\in[0,T), n\geq 1. \tag{2.43}$$

By the definitions of R_t and $\widetilde{W}_t$, we have for every $s \le t \wedge \zeta_n$,

$$\begin{aligned}\log R_s &= -\int_0^s h(r)\mathrm{d}\widetilde{W}_r + \frac{1}{2}\int_0^s |h(r)|^2 \mathrm{d}r \\ &\le -\int_0^s h(r)\mathrm{d}\widetilde{W}_r + C\left(1 + \int_0^s \frac{L^2|Y_r^{x,y}|^2}{|T-r|^2}\mathrm{d}r\right)\end{aligned}$$

And then, taking the expectation, we obtain with (2.43):

$$\mathbb{E}R_{t\wedge\zeta_n}\log R_{t\wedge\zeta_n} = \mathbb{E}^{t,n}\log R_{t\wedge\zeta_n} \le \frac{|x-y|^2}{T} + \frac{C}{\varepsilon}T, \qquad t \in [0,T), n \ge 1.$$

By the martingale convergence theorem and the Fatou lemma, $(R_{s\wedge\zeta})_{s\in[0,T]}$ is a well-defined martingale with

$$\mathbb{E}R_{t\wedge\zeta}\log R_{t\wedge\zeta} \le \frac{|x-y|^2}{T} + \frac{C}{\varepsilon}T, \qquad t \in [0,T].$$

To see that $(R_{s\wedge\zeta})_{s\in[0,T]}$ is a martingale, let $0 \le s < t \le T$. By the dominated convergence theorem and the martingale property of $(R_{t\wedge\zeta_n\wedge(T-1/n)})$, we have

$$\begin{aligned}\mathbb{E}(R_{t\wedge\zeta}|\mathscr{F}_s) &= \mathbb{E}(\lim_{n\to+\infty} R_{t\wedge\zeta_n\wedge(T-1/n)}|\mathscr{F}_s) = \lim_{n\to+\infty}\mathbb{E}(R_{t\wedge\zeta_n\wedge(T-1/n)}|\mathscr{F}_s) \\ &= \lim_{n\to+\infty} R_{s\wedge\zeta_n} = R_{s\wedge\zeta}.\end{aligned}$$

(2)Since $\widetilde{W}_t$ is a standard Brownian motion up to $T\wedge\zeta$, it follows from (2.42) that

$$\frac{n^2}{T}\mathbb{Q}(\zeta_n \le t) \le \mathbb{E}^{\mathbb{Q}}\frac{|Y_{t\wedge\zeta_n}^{x,y}|^2}{T-(t\wedge\zeta_n)} \le \frac{|x-y|^2}{T} + \frac{C}{\varepsilon}T$$

holds for $n \ge 1$ and $t \in [0,T)$. By letting $n \nearrow +\infty$, we obtain $\mathbb{Q}(\zeta \le t) = 0$ for all $t \in [0,T)$. This is equivalent to $\mathbb{Q}(\zeta = T) = 1$ according to the definition of ζ.
(3) Let

$$\overline{\zeta} = \inf\{t \in [0,T] : Y_t^{x,y} = 0\}$$

and set $\inf\varnothing = +\infty$ by convention. We claim that $\overline{\zeta} \le T$ and thus, $Y_T^{x,y} = 0$, $\mathbb{Q}$-a.s. Indeed, if for some $w \in \Omega$ such that $\overline{\zeta} > T$, by the continuity of the process we have

$$\inf_{t\in[0,T]} |Y_t^{x,y}|^2(w) > 0.$$

So,

$$\int_0^T \frac{|Y_s^{x,y}|^2}{|T-s|^2}\mathrm{d}s = \infty$$

holds on the set $\{\overline{\zeta} > T\}$. Now, since inequality (2.42) still hold with ζ_n replaced by ζ and since $\zeta = T$ $\mathbb{Q}$-a.s., if we take the expectation with respect to $\mathbb{Q}$, then

$$\mathbb{E}^{\mathbb{Q}}\int_0^T \frac{|Y_s^{x,y}|^2}{|T-s|^2}\mathrm{d}s \le \frac{|x-y|^2}{T} + \frac{C}{\varepsilon}T < +\infty.$$

Therefore $\mathbb{Q}(\overline{\zeta} > T) = 0$. Therefore, $Y_T^{x,y} = 0$, $\mathbb{Q}$-a.s. $\square$

Hence the can apply the Girsanov theorem. Therefore, there exist a new probability measure $\mathbb{Q}$ under which $\widetilde{W}$ is a Brownian motion. Thus, under $\mathbb{Q}$, $\widetilde{V}$ has the law μ^x whereas under $\mathbb{P}$ it has the law $\widetilde{\mu}$. Of course μ^x and $\widetilde{\mu}$ are equivalent. We deduce that for every $\delta > 0$,

$$\begin{aligned}
\int_{\mathscr{C}([0,T],\mathbb{R}^d)} \left(\frac{\mathrm{d}\mu^x}{\mathrm{d}\widetilde{\mu}}\right)^{2+\delta} \mathrm{d}\widetilde{\mu} &= \int_\Omega \left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)^{2+\delta} \mathrm{d}\mathbb{P} \\
&= \int_\Omega \left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)^{1+\delta} \mathrm{d}\mathbb{Q} \\
&= \mathbb{E}^{\mathbb{Q}} \left(\exp\left((1+\delta)I_T - \frac{1+\delta}{2}\langle I, I\rangle_T\right)\right) \\
&\leq \sqrt{\mathbb{E}^{\mathbb{Q}}\left(\exp(2(1+\delta)I_T - 2(1+\delta)^2)\langle I, I\rangle_T)\right)} \\
&\qquad \times \sqrt{\mathbb{E}^{\mathbb{Q}}\left(\exp((2(1+\delta)^2 - (1+\delta))\langle I, I\rangle_T)\right)} \\
&= \sqrt{\mathbb{E}^{\mathbb{Q}}\left(\exp((1+\delta)(1+2\delta)\langle I, I\rangle_T)\right)}. \qquad (2.44)
\end{aligned}$$

We are going to show that we can pick $L > 0$ and $\delta > 0$ such that:

$$\mathbb{E}^{\mathbb{Q}}\left(\exp((1+\delta)(1+2\delta)\langle I, I\rangle_T)\right) < +\infty.$$

Indeed, we have, for every $\iota > 0$

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}(\exp((1+\delta)(1+2\delta)\langle I, I\rangle_T)) \\
&\qquad \leq C\mathbb{E}^{\mathbb{Q}}\left(\exp\left((1+\delta)(1+2\delta)|||\sigma^{-1}|||^2\left((1+\iota)\int_0^T \frac{L^2|Y_s^{x,y}|^2}{|T-s|^2}\right)\mathrm{d}s\right)\right)
\end{aligned}$$

Now, by Itô's formula (using 2.41):

$$\begin{aligned}
\frac{L^2|Y_t^{x,y}|^2}{T-t} &= \frac{L^2|x-y|^2}{T} + 2\int_0^t \left(\frac{L^2 Y_s^{x,y}}{T-s}, \left(d(V_s^y + Y_s^{x,y}) - d(V_s^y) - \frac{LY_s^{x,y}}{T-s}\right)\mathrm{d}s\right) \\
&\quad - 2\int_0^t \left(\frac{L^2 Y_s^{x,y}}{T-s}, (\sigma(V_s^y + Y_s^{x,y}) - \sigma(V_s^y))(b(V_s^y + Y_s^{x,y}) - b(V_s^y))\mathrm{d}s\right) \\
&\quad + 2\int_0^t \left(\frac{L^2 Y_s^{x,y}}{T-s}, (\sigma(V_s^y + Y_s^{x,y}) - \sigma(V_s^y))\mathrm{d}\widetilde{W}_s\right) + \int_0^t \frac{L^2|Y_s^{x,y}|^2}{|T-s|^2}\mathrm{d}s \\
&\quad + \int_0^t \frac{L^2}{T-s}\operatorname{Tr}\left[[\sigma(V_s^y + Y_s^{x,y}) - \sigma(V_s^y)]^t[\sigma(V_s^y + Y_s^{x,y}) - \sigma(V_s^y)]\right]\mathrm{d}s.
\end{aligned}$$

Therefore, since d is dissipative, since b is bounded and since σ is Lipshitz and bounded,

$$\frac{L^2|Y_t^{x,y}|^2}{T-t} + 2\left(L - \frac{1}{2}\right)\int_0^t \frac{L^2|Y_s^{x,y}|^2}{|T-s|^2}\mathrm{d}s \leq \frac{L^2|x-y|^2}{T} + J_t + CL^2\int_0^t \frac{|Y_s^{x,y}|}{T-s}\mathrm{d}s$$

where $J_t = 2\int_0^t \left(\frac{L^2 Y_s^{x,y}}{T-s}, (\sigma(V_s^y + Y_s^{x,y}) - \sigma(V_s^y))\mathrm{d}\widetilde{W}_s\right)$. Note that

$$\begin{aligned}
\mathrm{d}\langle J, J\rangle_t &= 4L^4\frac{|^t(Y_s^{x,y})(\sigma(V_s^y + Y_s^{x,y}) - \sigma(V_s^y))|^2}{|T-s|^2}\mathrm{d}t \\
&\leq 4L^4\Lambda^2\frac{|Y_s^{x,y}|^2}{|T-s|^2}\mathrm{d}t.
\end{aligned}$$

Therefore, taking $t = T$, for every $\varepsilon > 0$, for every $\gamma > 0$, we have

$$2\left(L - \frac{1}{2} - \frac{\varepsilon}{4} - \gamma L^2\Lambda^2\right)\int_0^T \frac{L^2|Y_s^{x,y}|^2}{|T-s|^2}\mathrm{d}s \leq \frac{L^2|x-y|^2}{T} + \frac{C^2}{2\varepsilon}T + J_T - \frac{\gamma}{2}\langle J, J\rangle_T.$$

Therefore, for ε small enough, we obtain by multiplying by γ

$$2\gamma\left(L-\frac{1}{2}-\frac{\varepsilon}{4}-\gamma L^2\Lambda^2\right)\int_0^T\frac{L^2|Y_s^{x,y}|^2}{|T-s|^2}\mathrm{d}s\leq\frac{\gamma L^2|x-y|^2}{T}+\frac{\gamma C^2}{2\varepsilon}T+\gamma J_T-\frac{\gamma^2}{2}\langle J,J\rangle_T.$$

Therefore,

$$\mathbb{E}^{\mathbb{Q}}\left(\exp\left(2\gamma\left(L-\frac{1}{2}-\frac{\varepsilon}{4}-\gamma L^2\Lambda^2\right)\int_0^T\frac{L^2|Y_s^{x,y}|^2}{|T-s|^2}\mathrm{d}s\right)\right)\leq\exp\left(\frac{\gamma L^2|x-y|^2}{T}+\frac{\gamma C^2}{2\varepsilon}T\right).$$

Since by Hypothesis 2.1, there exists $\lambda>0$ such that

$$2(\lambda-\lambda^2\Lambda^2)>|||\sigma^{-1}|||^2,$$

it is enough to take $\iota>0$, $\varepsilon>0$, δ small enough, $\gamma L=\lambda$ and γ small enough such that:

$$2\left(\gamma L-\frac{\gamma}{2}-\frac{\gamma\varepsilon}{4}-\gamma^2L^2\Lambda^2\right)=2\left(\lambda-\frac{\gamma}{2}-\frac{\gamma\varepsilon}{4}-\lambda^2\Lambda^2\right)>(1+\delta)(1+2\delta)(1+\iota)|||\sigma^{-1}|||^2,$$

which shows that:

$$\mathbb{E}^{\mathbb{Q}}\left(\exp((1+\delta)(1+2\delta)\langle I,I\rangle_T)\right)<+\infty. \tag{2.45}$$

Therefore, with (2.44):

$$\int_{\mathscr{C}([0,T],\mathbb{R}^d)}\left(\frac{\mathrm{d}\widetilde{\mu}}{\mathrm{d}\mu^x}\right)^{2+\delta}\mathrm{d}\mu^x\leq\kappa_5. \tag{2.46}$$

We recall the following result (see for instance [58]).

Proposition 2.28. *Let (μ_1,μ_2) be two probability measures on a same space $(E,\mathscr{E})$ then*

$$||\mu_1-\mu_2||_{TV}=\min\mathbb{P}(Z_1\neq Z_2)$$

where the minimum is taken on all coupling (Z_1,Z_2) of (μ_1,μ_2). Moreover, there exists a coupling which realizes the infimum. We say that it is a maximal coupling. It satisfies

$$\mathbb{P}(Z_1=Z_2,Z_1\in\Gamma)=\mu_1\wedge\mu_2(\Gamma),\quad\Gamma\in\mathscr{B}(E).$$

Moreover, if μ_1 and μ_2 are equivalent and if

$$\int_E\left(\frac{\mathrm{d}\mu_2}{\mathrm{d}\mu_1}\right)^{p+1}\mathrm{d}\mu_1\leq C$$

for some $p>1$ and $C>1$ then

$$\mathbb{P}(Z_1=Z_2)=\mu_1\wedge\mu_2(E)\geq\left(1-\frac{1}{p}\right)\left(\frac{1}{pC}\right)^{1/(p-1)}.$$

Let us mention the following proposition which can be found in [65] under the name of Corollary 1.5.

Proposition 2.29. *Let E be a Polish space, $(\Omega,\mathscr{F},\mathbb{P})$ be a probability space and $(U_1,U_2,\widetilde{U})$ be three random variables on $(\Omega,\mathscr{F},\mathbb{P})$ taking value in E.*

Then there exists a triple $(u_1,u_2,\widetilde{u})$ such that $(u_2,\widetilde{u})$ is a maximal coupling of $(\mathscr{D}(U_2),\mathscr{D}(\widetilde{U}))$ and such that the law of $(u_1,\widetilde{u})$ is $\mathscr{D}(U_1,\widetilde{U})$.

Let us apply Proposition 2.29 to the three random variables $(Y^{x,y}, V^x, \widetilde{V})$: there exists $(\widehat{Y}^{x,y}, V^{1,x,y}, \widetilde{V}^{2,x,y})$ such that $(V^{1,x,y}, \widetilde{V}^{2,x,y})$ is a maximal coupling of $(\mu^x, \widetilde{\mu})$ on $[0,T]$ and $\mathscr{D}(\widehat{Y}^{x,y}, \widetilde{V}^{2,x,y}) = \mathscr{D}(Y^{x,y}, \widetilde{V})$. Therefore,

$$\mathscr{D}(\widetilde{V}^{2,x,y} - \widehat{Y}^{x,y}) = \mathscr{D}(\widetilde{V} - Y^{x,y}) = \mathscr{D}(V^y) := \mu^y,$$

and

$$\mathscr{D}(\widehat{Y}^{x,y}) = \mathscr{D}(Y^{x,y}).$$

Note that the last inequality implies that $\widehat{Y}^{x,y}_T = 0, \quad \mathbb{P}\text{-a.s.}$

Now remark that, $(V^{1,y,x}, V^{2,x,y} := \widetilde{V}^{2,x,y} - \widehat{Y}^{x,y})$ is a coupling of (μ^x, μ^y) and

$$\mathbb{P}(V^{1,y,x}_T = V^{2,x,y}_T) = \mathbb{P}(V^{1,y,x}_T = \widetilde{V}^{2,x,y}_T) \geq \mathbb{P}(V^{1,y,x} = \widetilde{V}^{2,x,y}).$$

Now, remarking that $(\widehat{V}^x, \widehat{V})$ is a maximal coupling of $(\mu^x, \widetilde{\mu})$ and applying Proposition 2.28 thanks to equation (2.46) we get that

$$\mathbb{P}(V^{1,y,x} = \widetilde{V}^{2,x,y}) \geq \frac{1}{4\kappa_5},$$

which leads to

$$\mathbb{P}(V^{1,y,x}_T = V^{2,x,y}_T) \geq \frac{1}{4\kappa_5}. \tag{2.47}$$

Step 3. We construct a coupling for any initial value and any date. For $x = y$, we set

$$(U^1_t, U^2_t) = (V^x_t, V^x_t), \quad t \in [0,T].$$

If x or y is not in B_R, then

$$(U^1_t, U^2_t) = (V^x_t, \overline{V}^y_t), \quad t \in [0,T]$$

where $\overline{V}^y_t$ is the solution of equation (2.7) driven by a Wiener process $\overline{W}$ independent of W. The coupling of the laws of V^x, V^y is defined as follows. Assuming that we have built (U^1_t, U^2_t) on $[0, nT]$, we take $(V^{1,x,y}, V^{2,x,y})$ as above independent on (U^1_t, U^2_t) on $[0, nT]$ and set

$$(U^1_{nT+t}, U^2_{nT+t}) = \left(V_t^{1,U^1_{nT},U^2_{nT}}, V_t^{2,U^1_{nT},U^2_{nT}}\right), \qquad \forall t \in [0,T].$$

The Markov property of (U^1, U^2) implies that (U^1, U^2) is a coupling of $(\mathscr{D}(V^x), \mathscr{D}(V^y))$ on $[0, (n+1)T]$.

We then define the following sequence of stopping times

$$L_m = \inf\{l > L_{m-1}, U^1_{lT}, U^2_{lT} \in B_R\}$$

with $L_0 = 0$. Evidently, (2.38) can be generalized to two solutions and we have:

$$\mathbb{E}(e^{\alpha L_1 T}) \leq \kappa_3(1 + |x|^2 + |y|^2)$$

and

$$\mathbb{E}\left(e^{\alpha(L_{m+1}-L_m)T})|\mathscr{F}_{L_m T}\right) \leq \kappa_3(1 + |U^1_{L_m T}|^2 + |U^2_{L_m T}|^2).$$

It follows that

$$\begin{aligned}\mathbb{E}\left(e^{\alpha L_{m+1}T}\right) &\leq \kappa_3 \mathbb{E}\left(e^{\alpha L_m T}(1+|U^1_{L_m T}|^2+|U^2_{L_m T}|^2)\right)\\ &\leq \kappa_3(1+2R^2)\mathbb{E}(e^{\alpha L_m T})\end{aligned}$$

and

$$\mathbb{E}\left(e^{\alpha L_m T}\right) \leq \kappa_3^m (1+2R^2)^{m-1}(1+|x|^2+|y|^2).$$

Now set

$$\ell_0 = \inf\{\ell, U^1_{(L_\ell+1)T} = U^2_{(L_\ell+1)T}\}.$$

Since $U^1_{L_\ell T}$, $U^2_{L_\ell T} \in B_R$, we have by (2.47),

$$\mathbb{P}(\ell_0 > \ell+1 | \ell_0 > \ell) \leq \left(1 - \frac{1}{\kappa_5}\right).$$

Since $\mathbb{P}(\ell_0 > \ell+1) = \mathbb{P}(\ell_0 > \ell+1|\ell_0 > \ell)\mathbb{P}(\ell_0 > \ell)$, we obtain

$$\mathbb{P}(\ell_0 > \ell) \leq \left(1 - \frac{1}{4\kappa_5}\right)^{\ell}.$$

Then for $\gamma \geq 0$

$$\begin{aligned}\mathbb{E}(e^{\gamma L_{\ell_0} T}) &\leq \sum_{\ell\geq 0} \mathbb{E}(e^{\gamma L_\ell T} \mathbb{1}_{\ell=\ell_0})\\ &\leq \sum_{\ell\geq 0} \mathbb{P}(\ell=\ell_0)^{1-\gamma/\alpha}(\mathbb{E}(e^{\alpha L_\ell T}))^{\gamma/\alpha}\\ &\leq \sum_{\ell\geq \ell_0} \left(1-\frac{1}{\kappa_5}\right)^{(\ell-1)(1-\gamma/\alpha)} [\kappa_3^\ell (1+2R^2)^{\ell-1}(1+|x|^2+|y|^2)]^{\gamma/\alpha}.\end{aligned}$$

We choose $\gamma \leq \alpha$ such that

$$\left(1-\frac{1}{4\kappa_5}\right)^{1-\gamma/\alpha} [\kappa_3(1+2R^2)]^{\gamma/\alpha} < 1$$

and deduce that

$$\mathbb{E}(e^{\gamma L_{\ell_0} T}) \leq \kappa_6(1+|x|^2+|y|^2).$$

Since

$$n_0 = \inf\left\{k, U^1_{kT} = U^2_{kT}\right\} \leq L_{\ell_0}+1$$

it follows that

$$\mathbb{E}(e^{\gamma n_0 T}) \leq \kappa_6(1+|x|^2+|y|^2)$$

and

$$\begin{aligned}\mathbb{P}(U^1_{kT} \neq U^2_{kT}) &= \mathbb{P}(k < n_0)\\ &\leq \kappa_6(1+|x|^2+|y|^2)e^{-\gamma k T}.\end{aligned}$$

Moreover, for all $t \in [0, T)$

$$\begin{aligned}\mathbb{P}(U^1_{kT+t} \neq U^2_{kT+t}) &\leq \mathbb{P}(U^1_{kT} \neq U^2_{kT})\\ &\leq \kappa_6(1+|x|^2+|y|^2)e^{-\gamma kT}\\ &\leq \kappa_7(1+|x|^2+|y|^2)e^{-\gamma(kT+t)}.\end{aligned}$$

We deduce for $\phi \in B_b(\mathbb{R}^d)$

$$|\mathscr{P}_t[\Phi](x) - \mathscr{P}_t[\Phi](y)| \leq C(1+|x|^2+|y|^2)e^{-\mu t}|\Phi|_0.$$

$\square$

2.5.3 Proof of Lemma 2.15

We follow the proof of the part 3 of [60]. We need to adapt this proof because in our case, the set in which the process is reflected is not bounded. Therefore convergences are not uniform in x anymore. In our case, the dissipativity of the process is enough to avoid the boundedness of $\overline{G}$. We will use the following notation $\beta(x) = (x - \Pi(x))$. Note that $F_n(x) = -2n\beta(x)$. We recall the following properties of the penalization term:

$$(x' - x, \beta(x)) \leq 0, \quad \forall x \in \mathbb{R}^d, \forall x \in \overline{G}, \tag{2.48}$$

$$(x' - x, \beta(x)) \leq (\beta(x'), \beta(x)), \quad \forall x, x' \in \mathbb{R}^n, \tag{2.49}$$

$$\exists c \in \overline{G}, \quad \gamma > 0, \quad \forall x \in \mathbb{R}^n, \quad (x - c, \beta(x)) \geq \gamma |\beta(x)|. \tag{2.50}$$

In what follows, C is a constant which may vary from line to line and which may depends on x and T but which is independent on n.

In a first time, we show that for any $1 \leq p < \infty$, there exists $C > 0$

$$\mathbb{E}\left[\left(n \int_0^T |\beta(X_t^{x,n})| \mathrm{d}s\right)^p\right] \leq C, \forall n \in \mathbb{N}. \tag{2.51}$$

For $p = 2$, Itô's formula gives us:

$$|X_t^{x,n} - c|^2 = |x - c|^2 + 2\int_0^t (X_s^{x,n} - c, f(X_s^{x,n})\mathrm{d}s + F_n(X_s^{x,n})\mathrm{d}s + \sigma(X_s^{x,n})\mathrm{d}W_s)$$
$$+ \int_0^t \sum_{i,j} (\sigma(X_s^{x,n})^t \sigma(X_s^{x,n}))_{i,j} \mathrm{d}s.$$

Using inequality (2.50), the fact that σ is bounded and Remark (2.3) we deduce:

$$4n\gamma \int_0^t |\beta(X_s^{x,n})| \mathrm{d}s \leq |x - c|^2 + 2\int_0^t C \mathrm{d}s + 2\int_0^t (X_s^{x,n} - c, \sigma(X_s^{x,n})\mathrm{d}W_s). \tag{2.52}$$

By a BDG inequality, using the fact that $|\sigma|$ is bounded:

$$\mathbb{E}\left[\left|\int_0^T (X_s^{x,n} - c, \sigma(X_s^{x,n})\mathrm{d}W_s)\right|^p\right] \leq C\mathbb{E}\left[\left(\int_0^T |X_s^{x,n} - c|^2 \mathrm{d}s\right)^{p/2}\right].$$

Now, by Lemma 2.1, it follows that the process $X^{x,n}$ has bounded moments of all orders independent of n, thus

$$\mathbb{E}\left[\left|\int_0^T (X_s^{x,n} - c, \sigma(X_s^{x,n})\mathrm{d}W_s)\right|^p\right] \leq C,$$

which leads to

$$\mathbb{E}\left[\left(n \int_0^T |\beta(X_t^{x,n})| \mathrm{d}s\right)^p\right] \leq C, \forall n \in \mathbb{N},$$

for a constant C which does not depend on n.

Now we prove that $\forall p > 2$,

$$\mathbb{E}\left(\sup_{0 \leq t \leq T} |\beta(X_t^{x,n})|^p\right) \leq \frac{C}{n^{p/2-1}}.$$

We apply Itô's formula to the function $\varphi(x) = |x - \Pi(x)|^p$ where $\beta(x) = x - \Pi(x)$. Note that $F_n(x) = -2n\beta(x)$. It is well known that for a regular boundary, for all $p > 2$, φ is $\mathscr{C}^2$ on $\mathbb{R}^d$ and $\nabla\varphi(x) = 2(x - \Pi(x))$. We recall the following formulas for the derivatives of φ,

$$\nabla\varphi(x) = p|\beta(x)|^{p-2}\beta(x),$$

$$\nabla^2\varphi(x) = p|\beta(x)|^{p-2}\nabla\beta(x) + p(p-2)|\beta(x)|^{p-4}(\beta(x)^t\beta(x)).$$

As $\nabla\beta(x)$ is a numerical matrix one can deduce the following inequality:

$$|\nabla^2\varphi(x)| \leq C|\beta(x)|^{p-2},$$

for a constant C which depends only on p and d.

We use Itô's formula, for all $p > 2$,

$$\begin{aligned}\varphi(X_t^{x,n}) = &\int_0^t (\nabla\varphi(X_s^{x,n}), d(X_s^{x,n}) + b(X_s^{x,n}) + F_n(X_s^{x,n}))\mathrm{d}s \\ &+ \int_0^t (\nabla\varphi(X_s^{x,n}), \sigma(X_s^{x,n}))\mathrm{d}W_s \\ &+ \frac{1}{2}\int_0^t \sum_{i,j}(\nabla^2\varphi(X_s^{x,n}))_{i,j}(\sigma(X_s^{x,n})^t\sigma(X_s^{x,n}))_{i,j}\mathrm{d}s.\end{aligned}$$

Therefore,

$$\begin{aligned}\varphi(X_t^{x,n}) + 2pn\int_0^t \varphi(X_s^{x,n})\mathrm{d}s \leq &\int_0^t (\nabla\varphi(X_s^{x,n}), d(X_s^{x,n}) + b(X_s^{x,n}))\mathrm{d}s \\ &+ \int_0^t (\nabla\varphi(X_s^{x,n}), \sigma(X_s^{x,n})\mathrm{d}W_s) + C\int_0^t |\beta(X_s^{x,n})|^{p-2}\mathrm{d}s.\end{aligned} \tag{2.53}$$

From Young's inequality: $ab \leq a^q/q + b^{q'}/q'$ for some real numbers q and q' such that $1/q + 1/q' = 1$, we choose $q = p/(p-2)$ and $q' = p/2$ so that, for $\alpha > 0$:

$$\begin{aligned}|\beta(X_s^{x,n})|^{p-2} &= \alpha n^{(p-2)/p}|\beta(X_s^{x,n})|^{p-2} \times \frac{1}{\alpha n^{(p-2)/p}} \\ &\leq \alpha^{p/(p-2)}\frac{p-2}{p}n|\beta(X_s^{x,n})|^p + \frac{2}{p}\left(\frac{1}{\alpha n^{(p-2)/p}}\right)^{p/2} \\ &\leq \alpha^{p/(p-2)}\frac{p-2}{p}n\varphi(X_s^{x,n}) + \frac{2}{p}\frac{1}{\alpha^{p/2}n^{(p-2)/2}},\end{aligned}$$

and another Young's inequality applied with this time $q = p/(p-1)$ and $q' = p$ gives us:

$$\begin{aligned}|(\nabla\varphi(X_s^{x,n}), d(X_s^{x,n}) + b(X_s^{x,n}))| &\leq p|\beta(X_s^{x,n})|^{p-1} \times |(d(X_s^{x,n}) + b(X_s^{x,n})| \\ &\leq \alpha^{p/(p-1)}n(p-1)p^{1/p}|\beta(X_s^{x,n})|^p \\ &\qquad + |d(X_s^{x,n}) + b(X_s^{x,n})|^p/(pn^{p-1}\alpha^p) \\ &\leq \alpha^{p/(p-1)}n(p-1)p^{1/p}\varphi(X_s^{x,n}) \\ &\qquad + (1 + |X_s^{x,n}|^{p\nu})/(pn^{p-1}\alpha^p).\end{aligned}$$

Therefore using the second inequality of Lemma 2.1 and the two above inequality we deduce, for α small enough:

$$\mathbb{E}\left(n\int_0^t \varphi(X_s^{x,n})\mathrm{d}s\right) \leq C\left(\frac{1}{n^{p-1}} + \frac{1}{n^{(p-2)/2}}\right)t,$$

therefore,

$$\mathbb{E}\left(\int_0^T \varphi(X_s^{x,n})\mathrm{d}s\right) \leq \frac{C}{n^{p/2}}. \tag{2.54}$$

Now we come back to equation (2.53). Taking the supremum over time and the expectation and using a BDG inequality we get:

$$\begin{aligned}\mathbb{E}\sup_{0\leq t\leq T}\varphi(X_t^{x,n}) &\leq \mathbb{E}\int_0^T |\nabla\varphi(X_s^{x,n})| \times |d(X_s^{x,n}) + b(X_s^{x,n})|\mathrm{d}s \\ &\quad + C\mathbb{E}\left[\left(\int_0^T |\nabla\varphi(X_s^{x,n})\sigma(X_s^{x,n})|^2\mathrm{d}s\right)^{1/2}\right] \\ &\quad + C\mathbb{E}\int_0^T |\beta(X_s^{x,n})|^{p-2}\mathrm{d}s. \end{aligned}\tag{2.55}$$

We call respectively I_1, I_2 and I_3 the three terms of the right hand side of (2.55). We have

$$\begin{aligned} I_1 &\leq C\sqrt{\mathbb{E}\int_0^T |\nabla\varphi(X_s^{x,n})|^2\mathrm{d}s} \times \sqrt{\mathbb{E}\int_0^T \left|1+|X_s^{x,n}|^{\nu}\right|^2\mathrm{d}s} \\ &\leq C\sqrt{\mathbb{E}\int_0^T |\beta(X_s^{x,n})|^{2p-2}\mathrm{d}s} \\ &\leq \frac{C}{n^{(p-1)/2}}, \end{aligned}$$

by using inequality (2.54) and Lemma 2.1. We also have

$$\begin{aligned} I_2 &\leq C\mathbb{E}\left[\left(\int_0^T |\beta(X_s^{x,n}|^{2p-2}\mathrm{d}s\right)^{1/2}\right] \\ &\leq \frac{1}{\sqrt{2}}\mathbb{E}\sup_{0\leq t\leq T}|\beta(X_s^{x,n})|^p + C'\mathbb{E}\left[\int_0^T |\beta(X_s^{x,n})|^{p-2}\mathrm{d}s\right], \end{aligned}$$

thanks to Young's inequality. Applying inequality (2.54) to the second member gives us:

$$I_2 \leq \frac{1}{\sqrt{2}}\mathbb{E}\sup_{0\leq t\leq T}|\beta(X_s^{x,n})|^p + \frac{C}{n^{(p-2)/2}}.$$

Finally applying inequality (2.54) once again gives us:

$$I_3 \leq \frac{C}{n^{(p-2)/2}}.$$

The above estimates of I_1, I_2 and I_3 give us the following inequality, for all $p > 2$

$$\mathbb{E}\sup_{0\leq t\leq T}\varphi(X_t^{x,n}) \leq \frac{C}{n^{(p-2)/2}}. \tag{2.56}$$

Now, we claim that for all $1 \leq p < +\infty$, $0 < 2q < p$, $n, m \in \mathbb{N}$,

$$\mathbb{E}\left[\left(m\int_0^T |{}^t\beta(X_s^{x,n})\beta(X_s^{x,m})|\mathrm{d}s\right)^p\right] \leq \frac{C}{n^q}. \tag{2.57}$$

Indeed, we have

$$m\int_0^T |{}^t\beta(X_s^{x,n})\beta(X_s^{x,m})|\mathrm{d}s \le AB$$
$$:= (\sup_{0\le s\le T}|\beta(X_s^{x,n})|)\left(m\int_0^T |\beta(X_s^{x,m})|\mathrm{d}s\right).$$

Since, for $r>2$

$$E\left[(AB)^p\right] \le (E(A^{rp}))^{1/r}(E(B^{r'p}))^{1/r'}, \qquad r'=\frac{r}{r-1},$$

from (2.51) and (2.56), we get

$$\mathbb{E}((AB)^p) \le \frac{C}{n^{(rp-2)/(2r)}} = \frac{C}{n^{p/2-1/r}},$$

which implies (2.57) for r large enough.

Now we will prove that if $2<2q<p<\infty$, there exists a constant C independent on n and m such that

$$\mathbb{E}\left[\sup_{0\le s\le T}|X_s^{x,n}-X_s^{x,m}|^p\right] \le C\left(\frac{1}{n}+\frac{1}{m}\right)^q, \qquad \forall n,m\in\mathbb{N}^*. \tag{2.58}$$

Indeed, applying Itô's formula, for all $0\le t\le T<+\infty$:

$$\begin{aligned}|X_t^{x,n}-X_t^{x,m}|^2 = {} & 2\int_0^t {}^t(X_s^{x,n}-X_s^{x,m})((d+b)(X_s^{x,n})-(d+b)(X_s^{x,m}))\mathrm{d}s\\ & -4n\int_0^t {}^t(X_s^{x,n}-X_s^{x,m})\beta(X_s^{x,n})\mathrm{d}s\\ & +4m\int_0^t {}^t(X_s^{x,n}-X_s^{x,m})\beta(X_s^{x,m})\mathrm{d}s\\ & +2\int_0^t {}^t(X_s^{x,n}-X_s^{x,m})(\sigma(X_s^{x,n})-\sigma(X_s^{x,m}))\mathrm{d}W_s\\ & +\int_0^t\sum_i[(\sigma(X_s^{x,n})-\sigma(X_s^{x,m}))^t(\sigma(X_s^{x,n})-\sigma(X_s^{x,m}))]_{i,i}\mathrm{d}s.\end{aligned}$$

By hypothesis on d, b and σ and thanks to equation (2.49) we obtain

$$\begin{aligned}|X_t^{x,n}-X_t^{x,m}|^2 \le {} & C\int_0^t |X_s^{x,n}-X_s^{x,m}|^2\mathrm{d}s\\ & +4\mathbb{E}n\int_0^t {}^t\beta(X_s^{x,n})\beta(X_s^{x,m})\mathrm{d}s + 4\mathbb{E}m\int_0^t {}^t\beta(X_s^{x,n})\beta(X_s^{x,m})\mathrm{d}s\\ & +2\int_0^t {}^t(X_s^{x,n}-X_s^{x,m})(\sigma(X_s^{x,n})-\sigma(X_s^{x,m}))\mathrm{d}W_s.\end{aligned} \tag{2.59}$$

Now, if I_t denotes the stochastic integral in (2.59) , we have for $r>1$

$$\mathbb{E}\left[\sup_{0\le s\le t}|I_s|^r\right] \le CE\left[\left(\int_0^t |X_s^{x,n}-X_s^{x,m}|^2\mathrm{d}s\right)^{r/2}\right].$$

Therefore, by virtue of property (2.57) and for a new constant C, $2r = p$, $q < p/2$ and $0 \le t \le T$, we deduce from (2.59)

$$\begin{aligned}\mathbb{E} \sup_{0\le s\le t} [|X_s^{x,n} - X_s^{x,m}|^p] &\le C \int_0^t E\,[|X_s^{x,n} - X_s^{x,m}|^p]\,\mathrm{d}s + C\left(\frac{1}{n^q} + \frac{1}{m^q}\right)\\ &\le C \int_0^t E \sup_{0\le u\le s} [|X_u^{x,n} - X_u^{x,m}|^p]\,\mathrm{d}s + C\left(\frac{1}{n^q} + \frac{1}{m^q}\right),\end{aligned}$$

which implies (2.58) after using Gronwall's Lemma. Inequality (2.58) shows us that the process $(X^{x,n})_n$ is Cauchy in $\mathscr{S}^p$, $p > 2$. Of course, taking the power $1/r$ for $r > 1$ in equation (2.58) shows us that $(X^{x,n})_n$ is Cauchy in $\mathscr{S}^p$, for all $p \ge 1$. Therefore we can define in $\mathscr{S}^p$, for all $p \ge 1$:

$$X^x := \lim_{n\to+\infty} X^{x,n}.$$

Furthermore, defining

$$\eta_t^{x,n} = -2n \int_0^t \beta(X_s^{x,n})\mathrm{d}s,$$

and remarking that the process $\eta^{x,n}$ satisfies the following equation

$$\eta_t^{x,n} = X_t^{x,n} - \int_0^t (d+b)(X_s^{x,n})\mathrm{d}s - \int_0^t \sigma(X_s^{x,n})\mathrm{d}W_s,$$

we can define in $\mathscr{S}^p$, for all $p \ge 1$:

$$\eta^x := \lim_{n\to+\infty} \eta^{x,n}.$$

Clearly, estimate (2.51) shows that η^x has locally bounded variation almost surely, and condition (2.56) implies that X^x belongs to $\overline{G}$ almost surely. Now remark that property (2.48) implies that, for all $T > 0$, for all progressively measurable process z taking values in the closed set $\overline{G}$

$$\int_0^T {}^t(X_s^{x,n} - z_s)\mathrm{d}\eta_s^{x,n} = -2n \int_0^T {}^t(X_s^{x,n} - z_s)\beta(X_s^{x,n})\mathrm{d}s \le 0. \tag{2.60}$$

Now we would like to pass to the limit into the previous inequality. For that purpose let us recall the following deterministic Lemma.

Lemma 2.30. *Let y_n be a sequence of functions in $\mathscr{C}([0,T],\mathbb{R}^k)$ which converges uniformly to y. Let η_n be a sequence of functions in $\mathscr{C}([0,T],\mathbb{R}^k)$ which converges uniformly to η and such that there exists $C > 0$ such that $||\eta_n||_{TV} \le C$. Then*

$$||\eta||_{TV} \le C \quad \textit{and} \quad \int_0^T y_n \mathrm{d}\eta_n \underset{n\to+\infty}{\longrightarrow} \int_0^T y\mathrm{d}\eta.$$

Proof. See Lemma 5.7 in [37]. □

Now we claim that we can extract a subsequence of $\left\{\int_0^T {}^t(X_s^{x,n} - z_s)\mathrm{d}\eta_s^{x,n}\right\}_n$ which converges $\mathbb{P}$-a.s. to $\int_0^T {}^t(X_s^x - z_s)\mathrm{d}\eta_s^x$. First, by equation (2.52), we have

$$2\gamma||\eta^{x,n}||_{TV}\mathrm{d}s \le |x-c|^2 + 2\int_0^t C\mathrm{d}s + 2\int_0^t (X_s^{x,n} - c, \sigma(X_s^{x,n})\mathrm{d}W_s).$$

Now remark that the right member of the inequality converges in probability to $|x-c|^2+2\int_0^t C\mathrm{d}s+2\int_0^t(X_s^x-c,\sigma(X_s^x)\mathrm{d}W_s)$. Therefore, there exists a subsequence $(\phi(n))_n$ such that the right member of the inequality converges $\mathbb{P}$-a.s. and then $||\eta^{x,\phi(n)}||_{TV}$ is bounded $\mathbb{P}$-a.s. Furthermore, as $\mathbb{E}\sup_{0\leq t\leq T}|X_t^{x,n}-X_t^x|^2 \underset{n\to+\infty}{\longrightarrow} 0$, and $\mathbb{E}\sup_{0\leq t\leq T}|\eta_t^{x,n}-\eta_t^x|^2 \underset{n\to+\infty}{\longrightarrow} 0$ we can extract a subsequence $(\psi(n))_n$ of $(\phi(n))_n$ such that $X^{x,\psi(n)}$ converge uniformly to X^x $\mathbb{P}$-a.s. and $\eta^{x,\psi(n)}$ converge uniformly to η^x $\mathbb{P}$-a.s. Now, it is enough to apply the previous deterministic Lemma $\mathbb{P}$-a.s. to obtain the result, namely, for all $T>0$, for all progressively measurable process z taking values in the closed set $\overline{G}$

$$\int_0^T {}^t(X_s^x-z_s)\mathrm{d}\eta_s^x \leq 0.$$

By Lemma 2.1 in [37], this implies that

$$\int_0^T \mathbb{1}_{\{X_s^x\in G\}}\mathrm{d}\eta_s^x=0, \qquad \mathbb{P}\text{-a.s.}$$

and

$$\eta_t^x=\int_0^t \nabla\phi(X_s^x)\mathrm{d}||\eta^x||_{TV,s}.$$

We define $\mathrm{d}K_s:=\mathrm{d}||\eta^x||_{TV,s}$. This shows that (X^x,K^x) satisfies the SDE (2.21).

Now we prove that the solution of the SDE (2.21) is unique. Let us assume that (X^1,K^1) and (X^2,K^2) are two solutions of (2.21) such that K^1 and K^2 have bounded variation on $[0,T]$, for all $T>0$ and such that for all $i=1,2$, for all continuous and progressively measurable process z taking values in the closure $\overline{G}$ we have,

$$\int_0^T {}^t(X_s^i-z_s)\mathrm{d}\eta_s^i \leq 0,$$

where $\eta_t^i=\int_0^t\nabla\phi(X_s^i)\mathrm{d}K_s^i$. Applying Itô's formula, one has

$$\begin{aligned}|X_t^1-X_t^2|^2 =&2\int_0^t {}^t(X_s^1-X_s^2)(d(X_s^1)-d(X_s^2))\mathrm{d}s+2\int_0^t {}^t(X_s^1-X_s^2)(b(X_s^1)-b(X_s^2))\mathrm{d}s\\ &+2\int_0^t {}^t(X_s^1-X_s^2)(\mathrm{d}\eta_s^1-\mathrm{d}\eta_s^2)\mathrm{d}s+2\int_0^t {}^t(X_s^1-X_s^2)(\sigma(X_s^1)-\sigma(X_s^2))\mathrm{d}W_s\\ &+\int_0^t \mathrm{Tr}[(\sigma(X_s^1)-\sigma(X_s^2)){}^t(\sigma(X_s^1)-\sigma(X_s^2))]\mathrm{d}s.\end{aligned}$$

Since, $2\int_0^t {}^t(X_s^1-X_s^2)(\mathrm{d}\eta_s^1-\mathrm{d}\eta_s^2)\mathrm{d}s\leq 0$, we easily get

$$\begin{aligned}\mathbb{E}\left[\sup_{0\leq t\leq T}|X_t^1-X_t^2|^2\right]&\leq C\mathbb{E}\left[\int_0^T|X_s^1-X_s^2|^2\mathrm{d}s\right]\\ &\leq C\left[\int_0^T\mathbb{E}\sup_{0\leq u\leq s}|X_u^1-X_u^2|^2\mathrm{d}s\right].\end{aligned}$$

Applying Gronwall's lemma allows us to conclude. □

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Chapter 3

Résultat complémentaire sur l'unicité des solutions

In this chaper we establish a uniqueness result for viscosity solutions of EBSDEs when Neumann boundary conditions are null for possibly unbounded convex set $\overline{G}$. Uniqueness for solutions of EBSDEs with non-zero Neumann boundary conditions is established in Chapter 5 when σ is constant.

3.1 Introduction

Let us come back to the uniqueness problem encountered in the previous chapter. We consider the reflected SDE, $\forall s \geq t$:

$$\begin{cases} X_s^{t,x} = x + \int_t^s f(X_r^{t,x})\mathrm{d}r + \int_t^s \sigma(X_r^{t,x})\mathrm{d}W_s + \int_t^s \nabla\phi(X_r^{t,x})\mathrm{d}K_s^{t,x}, \\ K_s^{t,x} = \int_t^s \mathbb{1}_{\{X_r^x \in \partial G\}}\mathrm{d}K_r^{t,x}, \end{cases} \tag{3.1}$$

and the ergodic BSDE with zero Neumann boundary conditions

$$Y_t^x = Y_T^x + \int_t^T [\psi(X_s^x, Z_s^x) - \lambda]\mathrm{d}s - \int_t^T Z_s^x \mathrm{d}W_s. \tag{3.2}$$

We gather the hypotheses used in the previous chapter together for reader's convenience:

Hypothesis 3.1.

1. $f = d + b$ is weakly dissipative,
2. $d : \mathbb{R}^d \to \mathbb{R}^d$ is locally Lipschitz and have polynomial growth: there exists $\nu > 0$ such that for all $x \in \mathbb{R}^d$, $|d(x)| \leq C(1 + |x|^\nu)$,
3. $b : \mathbb{R}^d \to \mathbb{R}^d$ is Lipschitz,
4. σ is Lipschitz, invertible, and $|\sigma|$ and $|\sigma^{-1}|$ are bounded by σ_∞,
5. there exists $\Lambda \geq 0$ such that for every $x, y \in \mathbb{R}^d$,

   $$|\,(y, \sigma(x+y) - \sigma(x))\,| \leq \Lambda|y|$$

   and there exists $\lambda > 0$ such that

   $$2(\lambda - \lambda^2\Lambda^2) > |||\sigma^{-1}|||^2,$$

6. $\psi(\cdot, z)$ is continuous ,
7. $|\psi(x, 0)| \leq M_\psi$,
8. $|\psi(x, z) - \psi(x, z')| \leq M_\psi |z - z'|$,
9. G is an open convex set of $\mathbb{R}^d$,
10. There exists a function $\phi \in \mathscr{C}_b^2(\mathbb{R}^d)$ such that $G = \{\phi > 0\}$, $\partial G = \{\phi = 0\}$ and $|\nabla \phi(x)| = 1$, $\forall x \in \partial G$.

3.2 The perturbed forward SDE

Let us consider the following stochastic differential equation with values in $\mathbb{R}^d$:

$$\begin{cases} \mathrm{d}X_t = d(X_t)\mathrm{d}t + b(t, X_t)\mathrm{d}t + \sigma \mathrm{d}W_t, & t \geq 0, \\ X_0 = x \in \mathbb{R}^d. \end{cases} \tag{3.3}$$

We will assume the following about the coefficients of the SDE:

Hypothesis 3.2. 1. $d : \mathbb{R}^d \to \mathbb{R}^d$ is locally Lipschitz, strict dissipative (i.e. there exists $\eta > 0$ such that for every $x, y \in \mathbb{R}^d$, $< d(x) - d(y), x - y > \leq -\eta |x - y|^2$) and with polynomial growth (i.e. there exists $\mu > 0$ such that for every $x \in \mathbb{R}^d$, $|d(x)| \leq C(1 + |x|^\mu)$).

2. $b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ is bounded and measurable.
3. $\sigma \in \mathbb{R}^{d \times d}$ is invertible.

Definition 3.1. *We say that the SDE* (3.3) *admits a weak solution if there exists a new $\mathscr{F}$-Brownian motion $(\widehat{W}^x)_{t \geq 0}$ with respect to a new probability measure $\widehat{\mathbb{P}}$ (absolutely coninuous with respect to $\mathbb{P}$), and an $\mathscr{F}$-adapted process $(\widehat{X}^x)_{t \geq 0}$ with continuous trajectories for which 3.3 holds with $(W)_{t \geq 0}$ replaced by $(\widehat{W}^x)_{t \geq 0}$.*

Lemma 3.2. *Assume that Hypothesis 3.2 holds true and that $b(t, \cdot)$ is Lipschitz for every $t \geq 0$. Then for every $x \in \mathbb{R}^d$, equation*(3.3) *admits a unique strong solution, that is, an adapted $\mathbb{R}^d$-valued process denoted by X^x with continuous paths satisfying $\mathbb{P}$-a.s.,*

$$X_t^x = x + \int_0^t d(X_s^x)\mathrm{d}s + \int_0^t b(s, X_s^x)\mathrm{d}s + \int_0^t \sigma(X_s^x)\mathrm{d}W_s, \quad \forall t \geq 0.$$

Furthermore, we have the following estimate, $\forall p \geq 1$,

$$\mathbb{E}[|X_s^x|^p] \leq C(1 + |x|^p), \tag{3.4}$$

If F is only bounded and measurable then there exists a weak solution $(\widehat{X}, \widehat{W})$ and unicity in law holds. Furthermore, (3.4) *still hold (with respect to the new probability measure).*

Proof. For the first part of the lemma see [40], Theorem 3.3 in chapter 1 or [56], Theorem 3.5. Estimates (3.4) is a simple consequence of Ito's formula. Weak existence and unicity in law are a direct consequence of a Girsanov's transformation. □

We define the Kolmogorov semigroup associated to Eq. (3.3) as follows: $\forall \phi : \mathbb{R}^d \to \mathbb{R}$ measurable with polynomial growth

$$\mathscr{P}_t[\phi](x) = \mathbb{E}\phi(X_t^x).$$

Lemma 3.3 (Basic coupling estimate)**.** *Assume that Hypothesis 3.2 hold true and that* $\forall t \geq 0$, $b(t,\cdot)$ *is Lipschitz. Then there exists* $\hat{c} > 0$ *and* $\hat{\eta} > 0$ *such that for all* $\phi : \mathbb{R}^d \to \mathbb{R}$ *measurable and bounded (i.e.* $\exists C, \mu > 0$ *such that* $\forall x \in \mathbb{R}^d$, $\phi(x) \leq C(1+|x|^\mu)$*),*

$$|\mathscr{P}_t[\phi](x) - \mathscr{P}[\phi](y)| \leq \hat{c}e^{-\hat{\eta}t}. \tag{3.5}$$

We stress the fact that $\hat{c}$ *and* $\hat{\eta}$ *depend on* b *only through* $\sup_{t\geq 0}\sup_{x\mathbb{R}^d}|b(t,x)|$.

Proof. See [55]. □

Corollary 3.4. *Relation* (3.5) *can be extended to the case in which* b *is only bounded and measurable and for all* $t \geq 0$*, there exits a uniformly bounded sequence of Lipschitz functions in* x, $(b_n(t,\cdot))_{n\geq 1}$ *(i.e.* $\forall t \geq 0, \forall n \in \mathbb{N}$, $b_n(t,\cdot)$ *is Lipschitz and* $\sup_n \sup_t \sup_x |b_n(t,x)| < +\infty$*) such that*

$$\lim_n b_n(t,x) = b(t,x), \qquad \forall t \geq 0, \forall x \in R^d.$$

Clearly in this case in the definition of $\mathscr{P}[\phi]$ *the mean value is taken with respect to the new probability measure* $\widehat{\mathbb{P}}$.

Proof. It is enough to adapt the proof of Corollary 2.5 in [26]. The goal is to show that, if $\mathscr{P}^n$ denotes the Kolmogorov semigroup corresponding to equation 3.3 but with b replaced by $\widetilde{b}_n$, then $\forall x \in \mathbb{R}^d$, $\forall t \geq 0$,

$$\mathscr{P}^n_t[\phi](x) \underset{n\to+\infty}{\longrightarrow} \mathscr{P}_t[\phi](x).$$

□

Remark 3.5. Similarly, if for every $t \geq 0$, there exits a uniformly bounded sequence of Lipschitz functions $(b_{m,n}(t,\cdot))_{m\in\mathbb{N},n\in\mathbb{N}}$ (i.e. $\forall t \geq 0, \forall n \in \mathbb{N}, \forall m \in \mathbb{N}$, $F_{m,n}(t,\cdot)$ is Lipschitz and $\sup_m \sup_n \sup_t \sup_x |b_{m,n}(t,x)| < +\infty$) such that

$$\lim_m \lim_n b_{m,n}(t,x) = b(t,x), \qquad \forall t \geq 0, \forall x \in \mathbb{R}^d,$$

then, if $\mathscr{P}^{m,n}$ is the Kolmogorov semigroup corresponding to equation 3.3 but with F replaced by $b_{m,n}$, we have $\forall t \geq 0$, $\forall x \in \mathbb{R}^d$,

$$\lim_m \lim_n \mathscr{P}^{m,n}_t[\phi](x) = \mathscr{P}[\phi](x),$$

which shows that relation (3.5) still hold.

We will need to apply the lemma above to some functions with particular form.

Lemma 3.6. *Let* $\psi : \mathbb{R}^d \times \mathbb{R}^{1\times k} \to \mathbb{R}$ *be continuous in the first variable and Lipschitz in the second one and* ζ, ζ' *be two continuous functions:* $\mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{1\times k}$ *be such that for all* $s \geq 0$, $\zeta(s,\cdot)$ *and* $\zeta'(s,\cdot)$ *are continuous. We define, for every* $s \geq 0$ *and* $x \in \mathbb{R}^d$,

$$\Upsilon(x) = \begin{cases} \frac{\psi(x,\zeta(s,x)) - \psi(x,\zeta'(s,x))}{|\zeta(s,x)-\zeta'(x)|^2} {}^t(\zeta(s,x) - \zeta'(s,x)), & \text{if } \zeta(s,x) \neq \zeta'(s,x), \\ 0, & \text{if } \zeta(s,x) = \zeta'(s,x). \end{cases}$$

Then for all $s \geq 0$*, there exists a uniformly bounded sequence of Lipschitz functions* $(\Upsilon_{m,n}(s,\cdot))_{m\in\mathbb{N},n\in\mathbb{N}}$ *(i.e., for every* $m \in \mathbb{N}^*$ *and* $n \in \mathbb{N}^*$, $\Upsilon_n(s,\cdot)$ *is Lipschitz and* $\sup_m \sup_n \sup_s \sup_x |\Upsilon(s,x)| < +\infty$*) such that for every* $s \geq 0$ *and for every* $x \in \mathbb{R}^d$,

$$\forall x \in \mathbb{R}^d, \lim_{m\to+\infty} \lim_{n\to+\infty} \Upsilon_{m,n}(s,x) = \Upsilon(s,x).$$

Proof. See the proof of Lemma 3.5 in [26]. □

3.3 Uniqueness result for Markovian solutions (Y, Z, λ) of EBSDE with zero-Neumann boundary conditions

Theorem 3.7. *Assume that Hypotheses 3.1 hold true, then the solution (Y^x, Z^x, λ) of the EBSDE (3.2) is unique in the class of solutions (Y, Z, λ) such that $Y = v(X^x)$, $v : \mathbb{R}^d \to \mathbb{R}$ is continuous, $|v(x)| \leq C(1+|x|^p)$ for some $p \geq 0$, $v(0) = 0$, $Z \in \mathscr{M}^2(\mathbb{R}_+, \mathbb{R}^{1\times d})$.*

Proof. Let $(Y^1 = v^1(X^x), Z^1, \lambda^1)$ and $(Y^2 = v^2(X^x), Z^2, \lambda^2)$ denote two solutions. Exactly like in Theorem (2.18), one can show that $\lambda^1 = \lambda^2 =: \lambda$. Let $(Y^{1,T,t,x}, Z^{1,T,t,x})$ be the solution of the following BSDE, $\forall s \in [t, T]$,

$$Y_s^{1,T,t,x} = v^1(X_T^{t,x}) + \int_t^T [\psi(X_s^x, Z_s^{1,T,t,x}) - \lambda]\mathrm{d}s - \int_t^T Z_s^{1,T,t,x}\mathrm{d}W_s.$$

By uniqueness of solutions to BSDE, we deduce that

$$v^1(x) = Y_0^{1,T,0,x}.$$

Now, we fix infinitely differentiable functions $\rho_\varepsilon : \mathbb{R}^d \to \mathbb{R}^+$ bounded together with their derivatives of all order, such that: $\int_{\mathbb{R}^d} \rho_\varepsilon(x)\mathrm{d}x = 1$ and

$$\mathrm{supp}(\rho_\varepsilon) \subset \left\{\xi \in \mathbb{R}^d : |\xi| \leq \varepsilon\right\}$$

where supp denotes the support. Then we define $\forall n \in \mathbb{N}$,

$$\begin{aligned}
(F_n)_\varepsilon(x) &= \int_{\mathbb{R}^d} \rho_\varepsilon(y)F_n(x-y)\mathrm{d}y,\\
d_\varepsilon(x) &= \int_{\mathbb{R}^d} \rho_\varepsilon(y)d(x-y)\mathrm{d}y,\\
b_\varepsilon(x) &= \int_{\mathbb{R}^d} \rho_\varepsilon(y)b(x-y)\mathrm{d}y,\\
\sigma_\varepsilon(x) &= \int_{\mathbb{R}^d} \rho_\varepsilon(y)\sigma(x-y)\mathrm{d}y.
\end{aligned}$$

Let us denote by $X^{t,x,n,\varepsilon}$ the solution of the following SDE, $\forall s \geq t$

$$X_s^{t,x,\varepsilon} = x + \int_t^s \left(d_\varepsilon + b_\varepsilon + (F_n)_\varepsilon\right)(X_r^{t,x,n,\varepsilon})\mathrm{d}r + \int_t^s \sigma_\varepsilon(X_r^{t,x,\varepsilon})\mathrm{d}W_r,$$

and let $(Y^{1,T,t,x,n,\varepsilon}, Z^{1,T,t,x,n,\varepsilon})$ be the solution of the following BSDE, $\forall s \in [t, T]$

$$Y_s^{1,T,t,x,n,\varepsilon} = v^1(X_s^{t,x,n,\varepsilon}) + \int_t^s [\psi(X_r^{t,x,n,\varepsilon}, Z_r^{1,T,t,x,n,\varepsilon}) - \lambda]\mathrm{d}r - \int_s^T Z_r^{1,T,t,x,n,\varepsilon}\mathrm{d}W_s.$$

Then by a stability result, (see for e.g. Lemma 2.3 of [11]), we deduce that

$$\lim_{n\to+\infty}\lim_{\varepsilon\to+\infty} Y_0^{1,T,0,x,n,\varepsilon} = Y_0^{1,T,0,x} = v^1(x). \tag{3.6}$$

Similarly, defining $(Y^{2,T,t,x}, Z^{2,T,t,x})$ and $(Y^{2,T,t,x,n,\varepsilon}, Z^{2,T,t,x,n,\varepsilon})$ in the same way, we deduce that

$$\lim_{n\to+\infty}\lim_{\varepsilon\to+\infty} Y_0^{2,T,0,x,n,\varepsilon} = Y_0^{2,T,0,x} = v^2(x) - v(x).$$

Furthermore, by Theorem 4.2 in [54] (note that it can be extended to the case in which the terminal condition and the generator is continuous in x and with polynomial growth in x exactly by the same arguments exposed in the Theorem 4.2 in [32], the only difference coming from the fact that the authors of this last paper work in infinite dimension for the SDE), if we define $u^{1,T,n,\varepsilon}(t,x) := Y_t^{1,T,t,x,n,\varepsilon}$, then $(x \mapsto u^{1,T,n,\varepsilon}(t,x))$ is continuously differentiable for all $t \in [0,T[$, and $\forall s \in [t,T[$,

$$Z_s^{1,T,t,x,n,\varepsilon} = {}^t\nabla u^{1,T,n,\varepsilon}(s, X_s^{t,x,n,\varepsilon})\sigma_\varepsilon(X_s^{t,x,n,\varepsilon}).$$

Similarly, we define $u^{2,T,n,\varepsilon}(t,x) := Y_t^{2,T,t,x,n,\varepsilon}$ and then

$$Z_s^{2,T,t,x,n,\varepsilon} = {}^t\nabla u^{2,T,n,\varepsilon}(s, X_s^{t,x,n,\varepsilon})\sigma_\varepsilon(X_s^{t,x,n,\varepsilon}).$$

Therefore, taking $t = 0$ (and omitting the superscript $t = 0$), $\forall T > 0$,

$$\begin{aligned} u^{1,T,n,\varepsilon}(0,x) - u^{2,T,n,\varepsilon}(0,x) &= (v^1 - v^2)(X_T^{x,n,\varepsilon}) - \int_0^T (Z_s^{1,T,x,n,\varepsilon} - Z_s^{2,T,x,n,\varepsilon})\mathrm{d}W_s \\ &\quad + \int_0^T \left[\psi(X_s^{x,n,\varepsilon}, Z_s^{1,T,x,n,\varepsilon}) - \psi(X_s^{x,n,\varepsilon}, Z_s^{2,T,x,n,\varepsilon})\right] \mathrm{d}s \\ &= (v^1 - v^2)(X_T^{x,n,\varepsilon}) \\ &\quad - \int_0^T (Z_s^{1,T,x,n,\varepsilon} - Z_s^{2,T,x,n,\varepsilon})(-\beta(s, X_s^{x,n,\varepsilon})\mathrm{d}s + \mathrm{d}W_s), \end{aligned}$$

where

$$\beta^T(s,x) =$$

$$\begin{cases} \frac{\psi(x, {}^t\nabla u^{1,T,n,\varepsilon}(s, X_s^{x,n,\varepsilon})\sigma) - \psi(x, {}^t\nabla u^{2,T,n,\varepsilon}(s, X_s^{x,n,\varepsilon})\sigma(X_s^{x,n,\varepsilon}))}{|{}^t\nabla u^{1,T,n,\varepsilon}(s, X_s^{x,n,\varepsilon})\sigma(X_s^{x,n,\varepsilon}) - {}^t\nabla u^{2,T,n,\varepsilon}(s, X_s^{x,n,\varepsilon})\sigma(X_s^{x,n,\varepsilon})|^2} \mathbb{1}_{t<T}, & \\ & \text{if } \nabla u^{1,T,n,\varepsilon}(t,x) \neq \nabla u^{2,T,n,\varepsilon}(t,x), \\ 0, & \text{otherwise.} \end{cases}$$

The process β^T is progressively measurable and bounded, therefore, we can apply Girsanov's Theorem to obtain that there exits a new probability measure $\mathbb{Q}^T$ equivalent to $\mathbb{P}$ under which $(W_t - \int_0^t \beta(s, X_s^{x,n,\varepsilon})\mathrm{d}s)_{t\in[0,T]}$ is a Brownian motion. Therefore, denoting by $E^{\mathbb{Q}^T}$ the expectation with respect to the probability $\mathbb{Q}^T$,

$$\begin{aligned} u^{1,T,n,\varepsilon}(0,x) - u^{2,T,n,\varepsilon}(0,x) &= \mathbb{E}^{\mathbb{Q}^T}\left[(v^1 - v^2)(X_T^{x,n\varepsilon})\right] \\ &= \mathscr{P}_T[v^1 - v^2](x), \end{aligned}$$

where $\mathscr{P}_t$ is the Kolmogorov semigroup associated to the following SDE, $\forall t \geq 0$,

$$U_t^x = x + \int_0^t (d_\varepsilon + b_\varepsilon + (F_n)_\varepsilon)(U_s^x)\mathrm{d}s + \int_0^t \sigma(U_s^x)\beta(s, U_s^x)\mathrm{d}s + \int_t^s \sigma(U_s^x)\mathrm{d}W_s.$$

By Corollary 3.4

$$|u^{1,T,n,\varepsilon}(0,x) - u^{2,T,n,\varepsilon}(0,x) - (u^{1,T,n,\varepsilon}(0,0) - (u^{2,T,n,\varepsilon}(0,0))| \leq Ce^{-\hat{\eta}T}.$$

Therefore, thanks to (3.6),

$$|v^1(x) - v^2(x) - (v^1(0) - v^2(0))| \leq Ce^{-\hat{\eta}T}.$$

Therefore, since $v^1(0) = v^2(0) = 0$, letting $T \to +\infty$ implies that

$$v^1(x) = v^2(x), \forall x \in \overline{G}.$$

An Itô's formula applied to $|Y_T^1 - Y_T^2|^2$ is enough to show that $\mathbb{E}\int_0^T |Z_s^1 - Z_S^2|\mathrm{d}s = 0$. $\square$

Remark 3.8. If the Neumann conditions are non-zero, it is possible to adapt the arguments used hereabove. Indeed, if $(Y^x = v(X^x), Z^x, \lambda)$ is a Markovian solution of the EBSDE, $\forall 0 \leq t \leq T < +\infty$,

$$Y_t^x = Y_T^x + \int_t^T [\psi(X_s^x, Z_s^x) - \lambda]\mathrm{d}s + \int_t^T g(X_s^x)\mathrm{d}K_s^x - \int_t^T Z_s \mathrm{d}W_s,$$

then, if $g \in \mathscr{C}^1_{\mathrm{lip}}(\overline{G})$, then as in the proof of Theorem 3.1 in [72], we deduce the existence of a function $\widetilde{v} : \overline{G} \to \mathbb{R}$ which belongs to the space $\mathscr{C}^2_{\mathrm{lip}}(\overline{G})$ and which is solution of the Helmholtz's equation for some $\alpha \in \mathbb{R}$,

$$\begin{cases} \Delta\widetilde{v}(x) - \alpha\widetilde{v}(x) = 0, \\ \frac{\partial\widetilde{v}(x)}{\partial n} + g(x) = 0 \end{cases}$$

We set $\widetilde{Y}_s^x = \widetilde{v}(X_s^x)$ and $\widetilde{Z}_s^x = \nabla\widetilde{v}(X_s^x)\sigma(X_s^x)$. These processes satisfies the EBSDE, $\forall 0 \leq t \leq T < +\infty$,

$$\widetilde{Y}_t^x = \widetilde{v}(X_T^x) + \int_t^T [-\mathscr{L}\widetilde{v}(X_r^x)]\mathrm{d}r + \int_t^T g(X_r^x)\mathrm{d}K_r^x - \int_t^T \widetilde{Z}_r^x \mathrm{d}W_r,$$

where

$$(\mathscr{L}\widetilde{v}(x)) = \frac{1}{2}\operatorname{Tr}(\sigma(x)^t\sigma(x)\nabla^2\widetilde{v}(x)) + \langle f(x), \nabla\widetilde{v}(x)\rangle.$$

Then, if we define

$$\begin{aligned} \overline{Y}_s^x &= Y_s^x - \widetilde{v}(X_s^x) \\ \overline{Z}_s^x &= Z_s^x - \nabla\widetilde{v}(X_s^x)\sigma(X_s^x), \end{aligned}$$

$(\overline{Y}^x, \overline{Z}^x)$ satisfies the EBSDE, $\forall 0 \leq t \leq T < +\infty$,

$$\begin{aligned} \overline{Y}_t^x &= \overline{Y}_T^x + \int_t^T \left[\psi(X_r^x, \overline{Z}_r^x + \nabla\widetilde{v}(X_r^x)\sigma(X_r^x)) + \mathscr{L}(\widetilde{v}(X_r^x)) - \lambda\right]\mathrm{d}r - \int_t^T \overline{Z}_r^x \mathrm{d}W_r \\ &= \overline{Y}_T^x + \int_t^T \left[\overline{\psi}(X_s^x, \overline{Z_s^x}) - \lambda\right]\mathrm{d}r - \int_t^T \overline{Z}_r^x \mathrm{d}W_r, \end{aligned}$$

where $\overline{\psi}(x,z) = \psi(x, z + \nabla\widetilde{v}(x)\sigma(x)) + \mathscr{L}(\widetilde{v}(x))$. Note that $\overline{\psi}$ satisfies the same hypothesis as ψ. Thus, since $(\overline{Y}^x, \overline{Z}^x, \lambda)$ satisfies an EBSDE with zero-Neumann boundary condition,we deduce that uniqueness for $(\overline{Y}^x, \overline{Z}^x, \lambda)$holds under the same assumptions (provided that $g \in \mathscr{C}^1_{\mathrm{lip}}(\overline{G})$) by Theorem 3.7, which implies uniqueness for (Y^x, Z^x, λ).

Deuxième partie

Comportement en temps long des solutions d'EDP paraboliques semi-linéaires

Chapitre 4

Comportement en temps long des solutions mild d'EDP paraboliques semi-linéaires en dimension infinie

Résumé: Nous étudions le comportement en temps long des solutions mild d'équations de HJB en dimension infinie par une approche purement probabiliste. Pour cela, nous montrons que la solution d'une EDSR en horizon fini T prise à l'instant initial se comporte comme un terme linéaire en T shifté par la solution de l'EDSR ergodique associée prise à l'instant initial. De plus nous donnons une vitesse explicite de convergence, qui semble est nouvelle, au vu de l'état état actuel de nos connaissances.

Mots clés: Equations différentielles stochastiques rétrogrades; équations différentielles stochastiques rétrogrades ergodiques; équations de HJB en dimension infinie; comportement en temps long; solutions mild; opérateur de Orstein-Uhlenbeck.

Abstract: We study the large time behaviour of mild solutions of HJB equations in infinite dimension by a purely probabilistic approach. For that purpose, we show that the solution of a backward SDE in finite horizon T taken at initial time behaves like a linear term in T shifted with the solution of the associated ergodic backward SDE taken at initial time. Moreover we give an explicit rate of convergence, which seems to be new to our best knowledge.

Key words: Backward stochastic differential equations; Ergodic backward stochastic differential equations; HJB equations in infinite dimension; Large time behaviour; Mild solutions; Ornstein-Uhlenbeck operator.

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It has been written in collaboration with Ying Hu and Adrien Richou.

4.1 Introduction

We are concerned with the large time behavior of solutions of the Cauchy problem in an infinite dimensional real Hilbert space H:

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = \mathscr{L}u(t,x) + f(x, \nabla u(t,x)G) & \forall (t,x) \in \mathbb{R}_+ \times H, \\ u(0,x) = g(x) & \forall x \in H, \end{cases} \tag{4.1}$$

where $u : \mathbb{R}_+ \times H \to \mathbb{R}$ is the unknown function and $\mathscr{L}$ is the formal generator of the Kolmogorov semigroup $\mathscr{P}_t$ of an H-valued random process solution of the following Ornstein-Uhlenbeck SDE:

$$\begin{cases} \mathrm{d}X_t = (AX_t + F(X_t^x))\mathrm{d}t + G\mathrm{d}W_t & t \in \mathbb{R}_+, \\ X_0 = x & x \in H, \end{cases}$$

with W a Wiener process with values in another real Hilbert space Ξ, assumed to be separable, and G a linear operator from Ξ to H. We recall that (formally), $\forall h : H \to \mathbb{R}$,

$$(\mathscr{L}h)(x) = \frac{1}{2}\operatorname{Tr}(GG^*\nabla^2 h(x)) + \langle Ax + F(x), \nabla h(x)\rangle.$$

Our method uses only probabilistic arguments and can be described as follows.

First, let (v, λ) be the solution of the ergodic PDE:

$$\mathscr{L}v + f(x, \nabla v(x)G) - \lambda = 0 \quad \forall x \in H.$$

Then we have the following probabilistic representation. Let $(Y^{T,x}, Z^{T,x})$ be the solution of the backward SDE:

$$\begin{cases} \mathrm{d}Y_s^{T,x} = -f(X_s^x, Z_s^{T,x})\mathrm{d}s + Z_s^{T,x}\mathrm{d}W_s, \\ Y_T^{T,x} = g(X_T^x), \end{cases}$$

and let (Y, Z, λ) be the solution of the ergodic backward SDE:

$$\mathrm{d}Y_s = -(f(X_s^x, Z_s^x) - \lambda)\mathrm{d}s + Z_s^x\mathrm{d}W_s.$$

Then we get

$$\begin{cases} Y_s^{T,x} = u(T-s, X_s^x), \\ Y_s^x = v(X_s^x). \end{cases}$$

Finally, due to Girsanov transformations and the use of an important coupling estimate result, we deduce that there exists a constant $L \in \mathbb{R}$ such that $\forall x \in H$,

$$Y_0^{T,x} - \lambda T - Y_0^x \underset{T\to+\infty}{\longrightarrow} L,$$

i.e.,

$$u(T,x) - \lambda T - v(x) \underset{T\to+\infty}{\longrightarrow} L.$$

Our method not only uses purely probabilistic arguments but also gives a rate of convergence:

$$|u(T,x) - \lambda T - v(x) - L| \leq C(1 + |x|^{2+\mu})e^{-\hat{\eta}T}.$$

The constant μ appearing above is the polynomial growth power of $g(\cdot)$ and $f(\cdot, 0)$, while $\hat{\eta}$ is linked to the dissipative constant of A.

Large time behavior of solutions has been studied for various types of HJB equations of second order; see, e.g., [7], [35], [43], and [61]. In [7], a result in finite dimension is stated under periodic assumptions for f and a periodic and Lipschitz assumption for g. Furthermore, they assume that $f(x,z)$ is of linear growth in z and bounded in x. In [35], some results are stated in finite dimensional framework, under locally Hölder conditions for the coefficients. More precisely, they assume that $f(x,z) = H_1(z) - H_2(x)$ with H_1 a Lipschitz function and with locally Hölder conditions for H_2 and g. They also treat the case of H_1 locally Lipschitz but consequently need to assume that H_2 and g are Lipschitz. Furthermore, they only treat the Laplacian case, namely, they assume that $G = I_d$. No result on rate of convergence is given in that paper. In [43], the authors deal with the problem in finite dimension. They also only treat the Laplacian case and assume that $f(x,z)$ is a convex function of quadratic growth in z and of polynomial growth in x. No result on rate of convergence is given in this paper. Up to our best knowledge, the explicit rate of convergence only appears in Theorem 1.2 of [61] but in finite dimension and under periodic assumptions for $f(\cdot,z)$ and $g(\cdot)$. Furthermore, they only deal with the Laplacian case and they assume restrictive assumptions on f (i.e., there exists $0 < m < M$ such that $m < f(x,z) \leq M(1+|z|)$ and boundedness hypotheses about the partial derivatives of first and second order of f).

In this paper, we will assume that A is a dissipative operator, $G : \Xi \to H$ is an invertible and bounded operator, $g : H \to \mathbb{R}$ continuous with polynomial growth, and $f : H \times \Xi^* \to \mathbb{R}$ continuous with polynomial growth in the first variable and Lipschitz in the second variable.

The paper is organized as follows. In section 2, we introduce some notation. In section 3, we recall some results about existence and uniqueness results for solutions of an Ornstein-Ulhenbeck SDE, a general BSDE, and an EBSDE that will be useful for what follow in the paper. In section 4, we study the behavior of the solution of the BSDE taken at initial time when the horizon T of the BSDE increases. More precisely, first we are concerned with the path dependent framework, where a very general result can be stated. Then in the Markovian framework we obtain a more precise result for the behavior of solutions and a rate of convergence is given. In section 5, we apply our result to an optimal control problem.

4.2 Notation

We introduce some notation. Let E_1, E_2, and E_3 be real separable Hilbert spaces. The norm and the scalar product will be denoted by $|\cdot|$, $\langle\cdot,\cdot\rangle$, with subscripts if needed. $L(E_1,E_3)$ is the space of linear bounded operators $E_1 \to E_3$, with the operator norm, which is denoted by $|\cdot|_{L(E_1,E_3)}$. The domain of a linear (unbounded) operator A is denoted by $D(A)$. $L_2(E_1,E_3)$ denotes the space of Hilbert-Schmidt operators from E_1 to E_3, endowed with the Hilbert-Schmidt norm, which is denoted by $|\cdot|_{L_2(E_1,E_3)}$.

Given $\phi \in B_b(E_1)$, the space of bounded and measurable functions $\phi : E_1 \to \mathbb{R}$, we denote by $||\phi||_0 = \sup_{x\in E_1} |\phi(x)|$.

We say that a function $F : E_1 \to E_3$ belongs to the class $\mathscr{G}^1(E_1,E_3)$ if it is continuous, has a Gâteaux derivative $\nabla F(x) \in L(E_1,E_3)$ at any point $x \in E_1$, and for every $k \in E_1$, the mapping $x \mapsto \nabla F(x)k$ is continuous from E_1 to E_3. Similarly, we say that a function $F : E_1 \times E_2 \to E_3$ belongs to the class $\mathscr{G}^{1,0}(E_1 \times E_2, E_3)$ if it is continuous, Gâteaux differentiable with respect to the first variable on $E_1 \times E_2$, and $\nabla_x F : E_1 \times E_2 \to L(E_1,E_3)$ is strongly continuous. In connection with stochastic equations, the space $\mathscr{G}^1$ has been introduced in [33], to which we refer the reader for further properties.

Given a real and separable Hilbert space K and a probability space $(\Omega,\mathscr{F},\mathbb{P})$ with a filtration $\mathscr{F}_t$, we consider the following classes of stochastic processes:

1. $L^p_{\mathscr{P}}(\Omega, \mathscr{C}([0,T];K))$, $p \in [1,\infty)$, $T > 0$, is the space of predictable processes Y with continuous paths on $[0,T]$ such that
$$|Y|_{L^p_{\mathscr{P}}(\Omega,\mathscr{C}([0,T];K))} = \mathbb{E} \sup_{t\in[0,T]} |Y_t|^p_K < \infty.$$
2. $L^p_{\mathscr{P}}(\Omega, L^2([0,T];K))$, $p \in [1,\infty)$, $T > 0$, is the space of predictable processes Y on $[0,T]$ such that
$$|Y|_{L^p_{\mathscr{P}}(\Omega,L^2([0,T];K))} = \mathbb{E} \left(\int_0^T |Y_t|^2_K \mathrm{d}t \right)^{p/2} < \infty.$$
3. $L^2_{\mathscr{P},\mathrm{loc}}(\Omega, L^2([0,\infty);K))$ is the space of predictable processes Y on $[0,\infty)$ which belong to the space $L^2_{\mathscr{P}}(\Omega, L^2([0,T];K))$ for every $T > 0$. We define in the same way $L^p_{\mathscr{P},\mathrm{loc}}(\Omega, \mathscr{C}([0,\infty);K))$.

In the following, we consider a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and a cylindrical Wiener process denoted by $(W_t)_{t\geq 0}$ with values in Ξ, which is a real and separable Hilbert space. $(\mathscr{F}_t)_{t\geq 0}$ will denote the natural filtration of W augmented with the family of $\mathbb{P}$-null sets of $\mathscr{F}$. H denotes a real and separable Hilbert space in which the SDE will take values.

4.3 Preliminaries

We will need some results about the solution of the SDE when a perturbation term F is in the drift.

4.3.1 The perturbed forward SDE

Let us consider the following mild SDE for an unknown process $(X_t)_{t\geq 0}$ with values in H:
$$X_t = e^{tA}x + \int_0^t e^{(t-s)A} F(s, X_s)\mathrm{d}s + \int_0^t e^{(t-s)A} G\mathrm{d}W_s \quad \forall t \geq 0, \quad \mathbb{P}\text{-a.s.} \tag{4.2}$$
Let us introduce the following hypothesis.

Hypothesis 4.1. 1. A is an unbounded operator $A : D(A) \subset H \to H$ with $D(A)$ dense in H. We assume that A is dissipative and generates a stable C_0-semigroup $\{e^{tA}\}_{t\geq 0}$. By this we mean that there exist constants $\eta > 0$ and $M > 0$ such that
$$\langle Ax, x\rangle \leq -\eta|x|^2 \quad \forall x \in D(A); \quad |e^{tA}|_{L(H,H)} \leq Me^{-\eta t} \quad \forall t \geq 0.$$
2. For all $s > 0$, e^{sA} is a Hilbert-Schmidt operator. Moreover $|e^{sA}|_{L_2(H,H)} \leq Ms^{-\gamma}$ with $\gamma \in [0, 1/2)$.
3. $F : \mathbb{R}_+ \times H \to H$ is bounded and measurable.
4. G is a bounded linear operator in $L(\Xi, H)$.
5. G is invertible. We denote by G^{-1} its bounded inverse given by Banach's heorem.

Remark 4.1. Note that under the previous set of hypotheses, we immediately get that
$$\begin{aligned}
|e^{sA}G|^2_{L_2(\Xi,H)} &\leq |G|^2_{L(\Xi,H)} |e^{sA}|^2_{L_2(H,H)} \\
&\leq |G|^2_{L(\Xi,H)} |e^{\frac{s}{2}A}|^2_{L(H,H)} |e^{\frac{s}{2}A}|^2_{L_2(H,H)} \\
&\leq M^2 e^{-\eta s} \left(\frac{s}{2}\right)^{-2\gamma},
\end{aligned}$$
which shows that for every $s > 0$ and $x \in H$, $e^{sA}G \in L_2(\Xi, H)$, which can be used to control the stochastic integral over the time.

Definition 4.2. *We say that the SDE (4.2) admits a martingale solution if there exists a new $\mathscr{F}$-Wiener process $(\widehat{W}^x)_{t\geq 0}$ with respect to a new probability measure $\widehat{\mathbb{P}}$ (absolutely continuous with respect to $\mathbb{P}$) and an $\mathscr{F}$-adapted process $\widehat{X}^x$ with continuous trajectories for which* (4.2) *holds with W replaced by $\widehat{W}$.*

Lemma 4.3. *Assume that Hypothesis 4.1, 1-4 holds and that F is bounded and Lipschitz in x. Then for every $p \in [2, \infty)$, for every $T > 0$ there exists a unique process $X^x \in L^p_{\mathscr{P}}(\Omega, \mathscr{C}([0,T]; H))$ solution of (4.2). Moreover,*

$$\sup_{0\leq t<+\infty} \mathbb{E}|X_t^x|^p \leq C(1+|x|)^p, \tag{4.3}$$

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |X_t^x|^p\right] \leq C(1+T)(1+|x|^p), \tag{4.4}$$

for some constant C depending only on p, γ, M and $\sup_{t\geq 0}\sup_{x\in H}|F(t,x)|$.

If F is only bounded and measurable, then the solution to (4.2) *still exists but in the martingale sense. Moreover (4.3) and* (4.4) *still hold (with respect to the new probability). Finally such a martingale solution is unique in law.*

Proof. For the first part of the lemma see [23], Theorem 7.4. For the estimate (4.4) see Appendix A.1 in [26]. The end of the lemma is a simple consequence of the Girsanov theorem. We will now show the estimate (4.4). The ideas of this proof are adapted from [33], but under our assumptions we obtain an interesting bound depending polynomially on T. We have

$$\begin{aligned}
\sup_{0\leq t\leq T}|X_t^x|^p &\leq C\left(|x|^p + C\sup_{0\leq t\leq T}\left(\int_0^t e^{-\eta(t-s)}ds\right)^p + \sup_{0\leq t\leq T}\left|\int_0^t e^{(t-s)A}GdW_s\right|^p\right)\\
&\leq C\left(1+|x|^p + \sup_{0\leq t\leq T}\left|\int_0^t e^{(t-s)A}GdW_s\right|^p\right).
\end{aligned}$$

Let us introduce

$$c_\alpha^{-1} = \int_s^t (t-u)^{\alpha-1}(u-s)^{-\alpha}du$$

with $\alpha \in]1/p, 1/2-\gamma[$: we can assume that p is large enough and then for small p we will just use the Jensen inequality to obtain the result. Then, the classical factorization method gives us

$$\begin{aligned}
\int_0^t e^{(t-s)A}GdW_s &= c_\alpha\int_0^t\int_s^t (t-u)^{\alpha-1}(u-s)^{-\alpha}du e^{(t-s)A}GdW_s\\
&= c_\alpha\int_0^t (t-u)^{\alpha-1}\int_0^u (u-s)^{-\alpha}e^{(t-s)A}GdW_s du\\
&= c_\alpha\int_0^t (t-u)^{\alpha-1}e^{(t-u)A}Y_u du
\end{aligned}$$

with

$$Y_u = \int_0^u (u-s)^{-\alpha}e^{(u-s)A}GdW_s.$$

We apply Hölder's inequality to obtain, with q the conjugate exponent of p $\left(\text{i.e., } \frac{1}{p}+\frac{1}{q}=1\right)$,

$$\begin{aligned}
\left|\int_0^t e^{(t-s)A} G dW_s\right|^p &\leq C\left(\int_0^t (t-u)^{(\alpha-1)q} e^{-(t-u)\eta q} du\right)^{p/q}\left(\int_0^t |Y_u|^p du\right) \\
&\leq C\left(\int_0^t s^{(\alpha-1)q} e^{-\eta q s} ds\right)^{p/q}\left(\int_0^T |Y_u|^p du\right) \\
&\leq C\int_0^T |Y_u|^p du.
\end{aligned}$$

Thus we obtain, thanks to the Burkholder-Davis-Gundy inequality,

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\int_0^t e^{(t-s)A} G dW_s\right|^p\right] &\leq C\int_0^T \mathbb{E}\left[|Y_u|^p\right] du \\
&\leq CT \sup_{0\leq u\leq T} \mathbb{E}\left[|Y_u|^p\right] \\
&\leq CT \sup_{0\leq u\leq T}\left(\int_0^u (u-s)^{-2\alpha-2\gamma} e^{-(u-s)\eta} ds\right)^{p/2} \\
&\leq CT \sup_{0\leq u\leq T}\left(\int_0^u v^{-2\alpha-2\gamma} e^{-v\eta} dv\right)^{p/2} \\
&\leq CT.
\end{aligned}$$

□

We define the Kolmogorov semigroup associated to (4.2) as follows: $\forall \phi : H \to \mathbb{R}$ measurable with polynomial growth

$$\mathscr{P}_t[\phi](x) = \mathbb{E}\phi(X_t^x).$$

Lemma 4.4 (basic coupling estimates)**.** *Assume that Hypothesis 4.1 holds true and that F is a bounded and Lipschitz function. Then there exist $\hat{c} > 0$ and $\hat{\eta} > 0$ such that $\forall \phi : H \to \mathbb{R}$ measurable with polynomial growth (i.e. $\exists C, \mu > 0$ such that $\forall x \in H$, $|\phi(x)| \leq C(1+|x|^\mu)$), $\forall x, y \in H$,*

$$|\mathscr{P}_t[\phi](x) - \mathscr{P}_t[\phi](y)| \leq \hat{c}(1+|x|^{1+\mu}+|y|^{1+\mu})e^{-\hat{\eta}t}. \tag{4.5}$$

We stress the fact that $\hat{c}$ and $\hat{\eta}$ depend on F only through $\sup_{t\geq 0}\sup_{x\in H}|F(t,x)|$.

Proof. In the same manner as in the proof of Theorem 2.4 in [26], we obtain, for every $x, y \in H$,

$$\mathbb{P}(X_t^x \neq X_t^y) \leq \hat{c}(1+|x|^2+|y|^2)e^{-\tilde{\eta}t}.$$

Hence we obtain, for every $x, y \in H$ and $\phi : H \to \mathbb{R}$ measurable and such that $\forall x \in H$, $|\phi(x)| \leq C(1+|x|^\mu)$,

$$\begin{aligned}
|\mathscr{P}_t[\phi](x) - \mathscr{P}_t[\phi](y)| &\leq \sqrt{\mathbb{E}(|\phi(X_t^x)-\phi(X_t^y)|^2)}\sqrt{\mathbb{P}(X_t^x \neq X_t^y)} \\
&\leq C(1+|x|^\mu+|y|^\mu)(1+|x|+|y|)e^{-(\tilde{\eta}/2)t} \\
&\leq C(1+|x|^{1+\mu}+|y|^{1+\mu})e^{-\hat{\eta}t}.
\end{aligned}$$

□

Corollary 4.5. *Relation (4.5) can be extended to the case in which F is only bounded measurable, and $\forall t \geq 0$ there exists a uniformly bounded sequence of Lipschitz functions in x $(F_n(t,\cdot))_{n\geq 1}$ (i.e., $\forall t \geq 0, \forall n \in \mathbb{N}$, $F_n(t,\cdot)$ is Lipschitz and $\sup_n \sup_t \sup_x |F_n(t,x)| < +\infty$) such that*

$$\lim_n F_n(t,x) = F(t,x) \qquad \forall t \geq 0, \forall x \in H.$$

Clearly in this case in the definition of $\mathscr{P}_t[\phi]$ the mean value is taken with respect to the new probability $\widehat{\mathbb{P}}$.

Proof. It is enough to show that if $\mathscr{P}^n$ is the Kolmogorov semigroup corresponding to (4.2) but with F replaced by F_n, then $\forall x \in H$ and $\forall t \geq 0$,

$$\mathscr{P}^n_t[\phi](x) \underset{n\to+\infty}{\longrightarrow} \mathscr{P}_t[\phi](x).$$

See the proof of Corollary 2.5 in [26] for more details. □

Remark 4.6. Similarly, if for every $t \geq 0$, there exists a uniformly bounded sequence of Lipschitz functions $(F_{m,n}(t,\cdot)_{m\in\mathbb{N},n\in\mathbb{N}}$ (i.e., $\forall t \geq 0$, $\forall n \in \mathbb{N}$, $\forall m \in \mathbb{N}$, $F_{m,n}(t,\cdot)$ is Lipschitz and $\sup_m \sup_n \sup_t \sup_x |F_{m,n}(x)| < +\infty$) such that

$$\lim_n \lim_m F_{m,n}(t,x) = F(t,x) \quad \forall t \geq 0, \forall x \in H,$$

then, if $\mathscr{P}^{m,n}$ is the Kolmogorov semigroup corresponding to (4.2) but with F replaced by $F_{m,n}$, we have $\forall x \in H$ and $\forall t \geq 0$,

$$\lim_n \lim_m \mathscr{P}^{m,n}_t[\phi](x) = \mathscr{P}_t[\phi](x).$$

We will need to apply the lemma above to some functions with particular form.

Lemma 4.7. *Let $f : H \times \Xi^* \to \mathbb{R}$ be continuous in the first variable and Lipschitz in the second one and $\zeta, \zeta' : \mathbb{R}_+ \times H \to \Xi^*$ be such that $\forall s \geq 0$, $\zeta(s,\cdot)$ and $\zeta'(s,\cdot)$ are weakly* continuous. We define*

$$\Upsilon(s,x) = \begin{cases} \frac{f(x,\zeta(s,x))-f(x,\zeta'(s,x))}{|\zeta(s,x)-\zeta'(s,x)|^2}(\zeta(s,x)-\zeta'(s,x))^* & \text{if } \zeta(s,x) \neq \zeta'(s,x), \\ 0 & \text{if } \zeta(s,x) = \zeta'(s,x). \end{cases}$$

There exists a uniformly bounded sequence of Lipschitz functions $(\Upsilon_{m,n}(s,\cdot))_{m\in\mathbb{N}^*,n\in\mathbb{N}^*}$ (i.e. $\forall m \in \mathbb{N}^*, \forall n \in \mathbb{N}^*$, $\Upsilon_{m,n}(s,\cdot)$ is Lipschitz and $\sup_m \sup_n \sup_s \sup_x |\Upsilon_{m,n}(s,x)| < \infty$) such that

$$\lim_n \lim_m \Upsilon_{m,n}(s,x) = \Upsilon(s,x) \quad \forall s \geq 0, \forall x \in H.$$

Proof. See the proof of Lemma 3.5 in [26]. □

4.3.2 The BSDE

Let us fix $T > 0$ and let us consider the following BSDE in finite horizon for an unknown process $(Y^{T,t,x}_s, Z^{T,t,x}_s)_{s\in[t,T]}$ with values in $\mathbb{R}\times\Xi^*$:

$$Y^{T,t,x}_s = \xi^T + \int_s^T f(X^{t,x}_r, Z^{T,t,x}_r)\mathrm{d}r - \int_s^T Z^{T,t,x}_r \mathrm{d}W_r \quad \forall s \in [t,T], \tag{4.6}$$

where $(X^{t,x}_s)_{s\geq 0}$ is the mild solution of (4.2) starting from x at time $t \geq 0$. If $t = 0$, we use the standard notation $Y^{T,x}_s := Y^{T,0,x}_s$ and $Z^{T,x}_s := Z^{T,0,x}_s$.

We will assume the following assumptions.

Hypothesis 4.2 (path dependent case)**.** There exist $l > 0$, $\mu \geq 0$ such that the function $f : H \times \Xi^* \to \mathbb{R}$ and ξ^T satisfy the following:

1. $F : H \to H$ is a Lipschitz bounded function and belongs to the class $\mathscr{G}^1$,
2. ξ^T is an H valued random variable $\mathscr{F}_T$ measurable and there exists $\mu \geq 0$ such that $|\xi^T| \leq C(1 + \sup_{t \leq s \leq T} |X_s^x|^\mu)$,
3. $\forall x \in H$, $\forall z, z' \in \Xi^*$, $|f(x,z) - f(x,z')| \leq l|z - z'|$,
4. $f(\cdot, z)$ is continuous and $\forall x \in H$, $|f(x,0)| \leq C(1 + |x|^\mu)$.

Lemma 4.8. *Assume that Hypotheses 4.1 and 4.2 hold true, then there exists a unique solution* $(Y^{T,t,x}, Z^{T,t,x}) \in L^p_{\mathscr{P}}(\Omega, \mathscr{C}([t,T];\mathbb{R})) \times L^p_{\mathscr{P}}(\Omega, L^2([t,T];\Xi^*))$ $\forall p \geq 2$ *to the BSDE (4.6).*

Proof. See [33], Proposition 4.3. □

We recall here the link between solutions of such BSDEs and PDEs which will justify our probabilistic approach. For this purpose we will consider the following set of Markovian hypotheses. Note that this set of hypotheses is a particular case of Hypothesis 4.2.

Hypothesis 4.3 (Markovian case)**.** There exist $l > 0$, $\mu \geq 0$ such that the function $f : H \times \Xi^* \to \mathbb{R}$ and ξ^T satisfy the following:

1. $F : H \to H$ is a Lipschitz bounded function that belongs to the class $\mathscr{G}^1$;
2. $\xi^T = g(X_T^{t,x})$, where $g : H \to \mathbb{R}$ is continuous and have polynomial growth: $\forall x \in H$, $|g(x)| \leq C(1 + |x|^\mu)$;
3. $\forall x \in H$, $\forall z, z' \in \Xi^*$, $|f(x,z) - f(x,z')| \leq l|z - z'|$,
4. $f(\cdot, z)$ is continuous and $\forall x \in H$, $|f(x,0)| \leq C(1 + |x|^\mu)$.

We recall the concept of mild solution. We consider the HJB equation

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} + \mathscr{L}u(t,x) + f(x, \nabla u(t,x)G) = 0 & \forall (t,x) \in \mathbb{R}_+ \times H, \\ u(T,x) = g(x) & \forall x \in H, \end{cases} \tag{4.7}$$

where $\mathscr{L}u(t,x) = \frac{1}{2}\operatorname{Tr}(GG^*\nabla^2 u(t,x)) + \langle Ax + F(x), \nabla u(t,x)\rangle$. We can define the semigroup $(\mathscr{P}_t)_{t\geq 0}$ corresponding to X by the formula $\mathscr{P}_t[\phi](x) = \mathbb{E}\phi(X_t^x)$ for all measurable functions $\phi : H \to \mathbb{R}$ having polynomial growth, and we notice that $\mathscr{L}$ is the formal generator of $\mathscr{P}_t$. We give the definition of a mild solution of (4.7).

Definition 4.9. *We say that a continuous function $u : [0,T] \times H \to \mathbb{R}$ is a mild solution of the HJB equation (4.7) if the following conditions hold:*

1. $u \in \mathscr{G}^{0,1}([0,T] \times H, \mathbb{R})$.
2. *There exist some constant $C > 0$ and some real function k satisfying $\int_0^T k(t)\mathrm{d}t < +\infty$ such that for all $x \in H$, $h \in H$, $t \in [0,T)$ we have*
$$|u(t,x)| \leq C(1 + |x|^C), \qquad |\nabla u(t,x)h| \leq C|h|k(t)(1 + |x|^C).$$
3. *The following equality holds:*
$$u(t,x) = \mathscr{P}_{T-t}[g](x) + \int_t^T \mathscr{P}_{s-t}[f(\cdot, \nabla u(t,\cdot)G)](x)\mathrm{d}s \qquad \forall t \in [0,T], \qquad \forall x \in H.$$

Lemma 4.10. *Assume that Hypotheses 4.1 and 4.3 hold true, then there exists a unique mild solution u of the HJB equation (4.7) given by the formula*

$$u_T(t,x) = Y_t^{T,t,x}.$$

Proof. See Theorem 4.2 in [32]. □

Remark 4.11. By the change of time $\widetilde{u}_T(t,x) := u_T(T-t,x)$, we remark that $\widetilde{u}_T(t,x)$ is the unique mild solution of (4.1). Now, remark that $\widetilde{u}_T(T,x) = u_T(0,x) = Y_0^{T,0,x} = Y_0^{T,x}$; therefore the large time behavior of $Y_0^{T,x}$ is the same as that of the solution of (4.1).

4.3.3 The EBSDE

Let us consider the following EBSDE for an unknown process $(Y_t^x, Z_t^x, \lambda)_{t\geq 0}$ with values in $\mathbb{R}\times\Xi^*\times\mathbb{R}$:

$$Y_t^x = Y_T^x + \int_t^T (f(X_s^x, Z_s^x) - \lambda)\mathrm{d}s - \int_t^T Z_s^x \mathrm{d}W_s \quad \forall 0 \leq t \leq T < +\infty. \tag{4.8}$$

Hypothesis 4.4. There exist $l > 0$, $\mu \geq 0$ such that the functions $F : H \to H$ and $f : H \times \Xi^* \to \mathbb{R}$ satisfy the following:

1. $F : H \to H$ is a Lipschitz bounded function and belongs to the class $\mathscr{G}^1$,
2. $\forall x \in H$, $\forall z, z' \in \Xi^*$, $|f(x,z) - f(x,z')| \leq l|z - z'|$,
3. $f(\cdot, z)$ is continuous and $\forall x \in H$, $|f(x,0)| \leq C(1 + |x|^\mu)$.

Lemma 4.12 (existence)**.** *Assume that Hypotheses 4.1 and 4.4 hold true; then there exists a solution* $(Y^x, Z^x, \lambda) \in L^2_{\mathscr{P},loc}(\Omega, \mathscr{C}([0,\infty[;\mathbb{R})) \times L^2_{\mathscr{P},loc}(\Omega, L^2([0,\infty[;\Xi^*)) \times \mathbb{R}$, *to the EBSDE (4.8). Moreover there exists* $v : H \to \mathbb{R}$ *of class* $\mathscr{G}^1$ *such that* $\forall x, x' \in H$, $\forall t \geq 0$,

$$\begin{aligned} &Y_t^x = v(X_t^x) \quad \text{and} \quad Z_t^x = \nabla v(X_t^x)G, \\ &v(0) = 0, \\ &|v(x) - v(x')| \leq C(1 + |x|^{1+\mu} + |x'|^{1+\mu}), \\ &|\nabla v(x)| \leq C(1 + |x|^{1+\mu}). \end{aligned}$$

Proof. This can be proved in the same way as in [26], the only difference coming from the polynomial growth of $f(x,0)$. □

Remark 4.13. We stress the fact that the method used for the construction of a solution to the EBSDE requires the generator f to have the invariance property $\forall(x,y,z) \in H\times\mathbb{R}\times\Xi^*, \forall c \in \mathbb{R}, f(x, y+c, z) = f(x,y,z)$, as well as to have $\forall x, y_1, y_2, z \in H \times \mathbb{R}^2 \times \Xi^*, \langle f(x,y_1,z) - f(x,y_2,z), y_1 - y_2\rangle \leq 0$. The first condition is equivalent to the fact that f does not depend on y which implies the second one.

Lemma 4.14 (uniqueness)**.** *The solution* (Y^x, Z^x, λ) *of previous lemma is unique in the class of solutions* (Y, Z, λ) *such that* $Y = v(X^x)$, $|v(x)| \leq C(1 + |x|^p)$ *for some* $p \geq 0$, $v(0) = 0$, $Z \in L^2_{\mathscr{P},loc}(\Omega, L^2([0,\infty);\Xi^*))$, *and* $Z = \zeta(X^x)$, *where* $\zeta : H \to \Xi^*$ *is continuous for the weak* topology.*

Proof. We give a simpler proof than that in [26]. Indeed, let us consider two solutions $(Y^1 = v^1(X^x), Z^1 = \zeta^1(X^x), \lambda^1)$ and $(Y^2 = v^2(X^x), Z^2 = \zeta^2(X^x), \lambda^2)$. From Theorem 3.10 in [26], we get $\lambda^1 = \lambda^2$. Then, we have

$$\begin{aligned} v^1(x) - v^2(x) &= v^1(X_T^x) - v^2(X_T^x) + \int_0^T (f(X_s^x, Z_s^1) - f(X_s^x, Z_s^2))\mathrm{d}s - \int_t^T (Z_s^1 - Z_s^2)\mathrm{d}W_s \\ &= v^1(X_T^x) - v^2(X_T^x) + \int_0^T (Z_s^1 - Z_s^2)\frac{(f(X_s^x, Z_s^1) - f(X_s^x, Z_s^2))(Z_s^1 - Z_s^2)^*}{|Z_s^1 - Z_s^2|^2}\mathrm{d}s \\ &\quad - \int_0^T (Z_s^1 - Z_s^2)\mathrm{d}W_s. \end{aligned}$$

Now we define

$$\beta(x) = \begin{cases} \frac{(f(x,\zeta^1(x))-f(x,\zeta^2(x)))(\zeta^1(x)-\zeta^2(x))^*}{|\zeta^1(x)-\zeta^2(x)|^2} & \text{if } \zeta^1(x)-\zeta^2(x) \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

As $\beta(X_s^x)$ is measurable and bounded, one can apply Girsanov's theorem to deduce the existence of a new probability $\mathbb{Q}^T$ under which $\widetilde{W}_t = W_t - \int_0^t \beta_s \mathrm{d}s$, $0 \leq s \leq T$ is a Wiener process. Then

$$\begin{aligned} v^1(x) - v^2(x) &= \mathbb{E}^{\mathbb{Q}^T}[v^1(X_T^x) - v^2(X_T^x)] \\ &= \mathscr{P}_T[v^1 - v^2](x), \end{aligned}$$

where $\mathscr{P}_t$ is the Kolmogorov semigroup associated to the following SDE:

$$\begin{cases} \mathrm{d}U_t^x = AU_t^x \mathrm{d}t + F(U_t^x)\mathrm{d}t + G\beta(U_t^x)\mathrm{d}t + G\mathrm{d}W_t, & t \geq 0, \\ U_0^x = x. \end{cases}$$

Now, note that β satisfies the hypotheses of Lemma 4.7; therefore by Remark 4.6,

$$\begin{aligned} |(v^1 - v^2)(x) - ((v^1 - v^2)(0))| &= \left|\mathscr{P}_T[v^1 - v^2](x) - \mathscr{P}_T[v^1 - v^2](0)\right| \\ &\leq C(1 + |x|^{p+1})e^{-\hat{\eta}T}. \end{aligned}$$

Then, letting $T \to +\infty$ and noting that $(v^1 - v^2)(0) = 0$ leads us to

$$v^1(x) = v^2(x) \quad \forall x \in H.$$

An Itô's formula applied to $|Y_t^1 - Y_t^2|^2$ is enough to show that $\forall T > 0$

$$\mathbb{E}\int_0^T |Z_s^1 - Z_s^2|^2 \mathrm{d}s = 0,$$

which concludes the proof of uniqueness. □

Similarly to the case of BSDE, we recall the link between solutions of such EBSDEs and ergodic HJB equations. We consider the following ergodic HJB equation for an unknown pair $(v(\cdot), \lambda)$:

$$\mathscr{L}v(x) + f(x, \nabla v(x)G) - \lambda = 0 \quad \forall x \in H. \tag{4.9}$$

Since we are dealing with an elliptic equation it is natural to consider (v, λ) as a mild solution of (4.9) if and only if, for arbitrary time $T > 0$, $v(x)$ coincides with the mild solution $u(t, x)$ of the corresponding parabolic equation having v as a terminal condition

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} + \mathscr{L}u(t,x) + f(x, \nabla u(t,x)G) - \lambda = 0, & \forall t \in [0,T], \quad \forall x \in H, \\ u(T,x) = v(x), \quad \forall x \in H. \end{cases}$$

Thus we are led to the following definition.

Definition 4.15. *A pair (v, λ), ($v : H \to \mathbb{R}$ and $\lambda \in \mathbb{R}$) is a mild solution of the HJB equation (4.9) if the following are satisfied:*

1. *$v \in \mathscr{G}^1(H, \mathbb{R})$;*
2. *there exists $C > 0$ such that $|\nabla v(x)| \leq C(1 + |x|^C)$ for every $x \in H$;*
3. *$\forall 0 \leq t \leq T$ and $x \in H$,*

$$v(x) = \mathscr{P}_{T-t}[v](x) + \int_t^T (\mathscr{P}_{s-t}[f(\cdot, \nabla v(\cdot)G)](x) - \lambda)\mathrm{d}s.$$

We recall the following result.

Lemma 4.16. *Assume that Hypotheses 4.1 and 4.4 hold true. Then (4.9) admits a unique mild solution which is the pair (v, λ) defined in Lemma 4.12.*

Proof. See Theorem 4.1 in [26]. □

4.4 Large time behavior

We recall that $(Y_s^{T,x}, Z_s^{T,x})_{s\geq 0}$ denotes the solution of the finite horizon BSDE (4.6) with $t=0$ and that (Y_s^x, Z_s^x, λ) denotes the solution of the EBSDE (4.8).

4.4.1 First behavior: Path dependent framework and Markovian framework

Theorem 4.17. *Assume that Hypotheses 4.1 and 4.2 hold true (path dependent case). Then,* $\forall T>0$, $\forall n\in\mathbb{N}^*$,

$$\left|\frac{Y_0^{T,x}}{T}-\lambda\right| \leq \frac{C_n(1+T^{1/n})(1+|x|^{1+\mu})}{T}. \tag{4.10}$$

In particular,

$$\frac{Y_0^{T,x}}{T} \underset{T\to+\infty}{\longrightarrow} \lambda$$

uniformly in any bounded set of H.

Assume that Hypotheses 4.1 and 4.3 hold true (Markovian case). Then, $\forall T>0$,

$$\left|\frac{Y_0^{T,x}}{T}-\lambda\right| \leq \frac{C(1+|x|^{1+\mu})}{T}, \tag{4.11}$$

i.e.,

$$\left|\frac{u(T,x)}{T}-\lambda\right| \leq \frac{C(1+|x|^{1+\mu})}{T}, \tag{4.12}$$

where u *is the mild solution of (4.1). In particular,*

$$\frac{u(T,x)}{T} = \frac{Y_0^{T,x}}{T} \underset{T\to+\infty}{\longrightarrow} \lambda$$

uniformly in any bounded set of H.

Proof. First we treat the path dependent case, that is, when Hypotheses 4.1 and 4.2 hold true. For all $x\in H$, $T>0$,

$$\left|\frac{Y_0^{T,x}}{T}-\lambda\right| \leq \left|\frac{Y_0^{T,x}-Y_0^x-\lambda T}{T}\right| + \left|\frac{Y_0^x}{T}\right|.$$

We have

$$\begin{aligned} Y_0^{T,x}-Y_0^x-\lambda T &= \xi^T - v(X_T^x) + \int_0^T (f(X_s^x, Z_s^{T,x}) - f(X_s^x, Z_s^x))\mathrm{d}s \\ &\quad - \int_0^T (Z_s^{T,x}-Z_s^x)\mathrm{d}W_s \\ &= \xi^T - v(X_T^x) + \int_0^T (Z_s^{T,x}-Z_s)\beta_s^T \mathrm{d}s - \int_0^T (Z_s^{T,x}-Z_s)\mathrm{d}W_s, \end{aligned}$$

where $\forall s\in[0,T]$

$$\beta_s^T = \begin{cases} \frac{(f(X_s^x,Z_s^{T,x})-f(X_s^x,Z_s^x))(Z_s^{T,x}-Z_s^x)^*}{|Z_s^{T,x}-Z_s^x|^2} & \text{if } Z_s^{T,x}-Z_s^x \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

The process β_s^T is progressively measurable and bounded; therefore we can apply Girsanov's theorem to obtain that there exists a probability measure $\mathbb{Q}^T$ under which $\widetilde{W}_t^T = -\int_0^t \beta_s^T \mathrm{d}s + W_t$, $0 \leq t \leq T$, is a Wiener process. We recall that if we define

$$M_T = \exp\left(\int_0^T \beta_s^T \mathrm{d}W_s - \frac{1}{2}\int_0^T |\beta_s^T|_\Xi^2 \mathrm{d}s\right),$$

the following formula holds: $\mathrm{d}\mathbb{Q}^T = M_T \mathrm{d}\mathbb{P}$.

Taking the expectation with respect to $\mathbb{Q}^T$ we get

$$Y_0^{T,x} - Y_0^x - \lambda T = \mathbb{E}^{\mathbb{Q}^T}(\xi^T - v(X_T^x)). \tag{4.13}$$

Hence we have

$$\left|\frac{Y_0^{T,x} - Y_0^x - \lambda T}{T}\right| \leq C\frac{1 + \mathbb{E}^{\mathbb{Q}^T}[\sup_{0\leq t\leq T}|X_t^x|^\mu]}{T} + C\frac{1 + \mathbb{E}^{\mathbb{Q}^T}\left(|X_T^x|^{1+\mu}\right)}{T}.$$

The process $(X_t^x)_{t\geq 0}$ is the mild solution of

$$\begin{cases} \mathrm{d}X_t^x = AX_t^x \mathrm{d}t + F(X_t^x)\mathrm{d}t + G\beta_t^T \mathrm{d}t + G\mathrm{d}\widetilde{W}_t^T, & t \in [0,T], \\ X_0^x = x. \end{cases}$$

Thus, by Jensen's inequality and the estimate (4.4), there exists a constant C_n which does not depend on time such that

$$\mathbb{E}^{\mathbb{Q}^T}[\sup_{0\leq t\leq T}|X_t^x|^\mu] \leq (\mathbb{E}^{\mathbb{Q}^T}[\sup_{0\leq t\leq T}|X_t^x|^{n\mu}])^{1/n} \leq C_n(1 + T^{1/n})(1 + |x|^\mu),$$

and by Lemma 4.3,

$$\mathbb{E}^{\mathbb{Q}^T}(|X_T^x|^{1+\mu}) \leq C(1 + |x|^{1+\mu}),$$

which allows us to obtain

$$\left|\frac{Y_0^{T,x} - Y_0^x - \lambda T}{T}\right| \leq \frac{C_n(1 + T^{1/n})(1 + |x|^{1+\mu})}{T}.$$

Finally we note that

$$\left|\frac{Y_0^x}{T}\right| \leq \frac{C(1 + |x|^{1+\mu})}{T},$$

which gives the result for the path dependent case.

Now we treat the Markovian case: by equality (4.13), we obtain

$$Y_0^{T,x} - Y_0^x - \lambda T = \mathbb{E}^{\mathbb{Q}^T}(g(X_T^x) - v(X_T^x)). \tag{4.14}$$

Therefore, since $|g(x) - v(x)| \leq C(1 + |x|^{1+\mu})$,

$$\left|\frac{Y_0^{T,x} - Y_0^x - \lambda T}{T}\right| \leq C\frac{1 + \mathbb{E}^{\mathbb{Q}^T}\left(|X_T^x|^{1+\mu}\right)}{T} \leq C\frac{1 + |x|^{1+\mu}}{T},$$

which gives the result. □

Remark 4.18. If G is possibly degenerate, Theorem 4.1 remains true under additional assumptions that f is locally Lipschitz in x (i.e. $\exists \mu \geq 0$, $\forall x, x' \in H$, $\forall z \in \Xi^*$, $|f(x,z) - f(x',z)| \leq C(1 + |x|^\mu + |x'|^\mu)|x - x'|$) and that $A + F$ is dissipative. In this case, we have existence of a solution to the EBSDE and λ is unique from [31].

4.4.2 Second behavior and third behavior: Markovian framework

In this section, we introduce a new hypothesis set without loss of generality. Note that it is the same as Hypothesis 4.3 but with $F \equiv 0$. However, we write it again for the reader's convenience.

Hypothesis 4.5 (Markovian case, $F \equiv 0$)**.** There exist $l > 0$, $\mu \geq 0$ such that the function $f : H \times \Xi^* \to \mathbb{R}$ and ξ^T satisfy

1. $F \equiv 0$;
2. $\xi^T = g(X_T^x)$, where $g : H \to \mathbb{R}$ is continuous and have polynomial growth: for all $x \in H$, $|g(x)| \leq C(1 + |x|^\mu)$;
3. $\forall x \in H$, $\forall z, z' \in \Xi^*$, $|f(x, z) - f(x, z')| \leq l|z - z'|$;
4. $f(\cdot, z)$ is continuous and of polynomial growth, i.e., $\forall x \in H$, $|f(x, 0)| \leq C(1+|x|^\mu)$.

Remark 4.19. Note that setting $F \equiv 0$ is not restrictive. Indeed let us recall that the purpose of this paper is to study the large time behavior of the mild solution of

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = \mathscr{L}u(t, x) + f(x, \nabla u(t, x)G) & \forall (t, x) \in \mathbb{R}_+ \times H, \\ u(0, x) = g(x) & \forall x \in H. \end{cases}$$

Now remark that

$$\langle Ax + F(x), \nabla u(t, x)\rangle + f(x, \nabla u(t, x)G) = \langle Ax, \nabla u(t, x)\rangle + \widetilde{f}(x, \nabla u(t, x)G),$$

where $\widetilde{f}(x, z) = f(x, z) + \langle F(x), zG^{-1}\rangle$ is a continuous function in x with polynomial growth in x and Lipschitz in z. Therefore, under our assumptions, we can always consider the case $F \equiv 0$ by replacing f by $\widetilde{f}$ if necessary.

Theorem 4.20. *Assume that Hypotheses 4.1 and 4.5 hold true. Then there exists $L \in \mathbb{R}$ such that*

$$\forall x \in H, \quad Y_0^{T,x} - \lambda T - Y_0^x \underset{T\to+\infty}{\longrightarrow} L,$$

i.e.,

$$\forall x \in H, \quad u(T, x) - \lambda T - v(x) \underset{T\to+\infty}{\longrightarrow} L,$$

where u is the mild solution of (4.1).

Furthermore the following rate of convergence holds:

$$|Y_0^{T,x} - \lambda T - Y_0^x - L| \leq C(1 + |x|^{2+\mu})e^{-\hat{\eta}T},$$

i.e.,

$$|u(T, x) - \lambda T - v(x) - L| \leq C(1 + |x|^{2+\mu})e^{-\hat{\eta}T},$$

where u is the mild solution of (4.1).

Proof. Let us start by defining

$$u_T(t, x) := Y_t^{T,t,x},$$
$$w_T(t, x) := u_T(t, x) - \lambda(T - t) - v(x).$$

We recall that $Y_s^{T,t,x} = u_T(s, X_s^{t,x})$ and that $Y_s^x = v(X_s^x)$, where v is defined in Lemma 4.8. We recall that $\forall T, S \geq 0$, u_T is the unique mild solution of

$$\begin{cases} \frac{\partial u_T(t,x)}{\partial t} + \mathscr{L} u_T(t,x) + f(x, \nabla u_T(t,x)G) = 0 & \forall (t,x) \in [0,T] \times H, \\ u_T(T,x) = g(x) & \forall x \in H \end{cases}$$

and that u_{T+S} is the unique mild solution of

$$\begin{cases} \frac{\partial u_{T+S}(t,x)}{\partial t} + \mathscr{L} u_{T+S}(t,x) + f(x, \nabla u_{T+S}(t,x)G) = 0 & \forall (t,x) \in [0,T+S] \times H, \\ u_{T+S}(T+S,x) = g(x) & \forall x \in H. \end{cases}$$

This implies that $u_T(0,x) = u_{T+S}(S,x)$, $\forall x \in H$, and then,

$$w_T(0,x) = w_{T+S}(S,x). \tag{4.15}$$

We will need some estimates on w_T given in the following lemma.

Lemma 4.21. *Under the hypotheses of Theorem 4.20, $\exists C > 0$, $\forall x, y \in H$, $\forall T > 0$, $\forall 0 < T' \leq T$, $\exists C_{T'} > 0$,*

$$\begin{aligned} |w_T(0,x)| &\leq C(1 + |x|^{1+\mu}), \\ |\nabla_x w_T(0,x)| &\leq \frac{C_{T'}}{\sqrt{T'}}(1 + |x|^{1+\mu}), \\ |w_T(0,x) - w_T(0,y)| &\leq C(1 + |x|^{2+\mu} + |y|^{2+\mu})e^{-\hat{\eta}T}. \end{aligned}$$

We stress the fact that C depends only on η, M, γ, G, μ, $\sup_{x \in H} \frac{|g(x)|}{1+|x|^\mu}$, $\sup_{x \in H} \frac{|f(x)|}{1+|x|^\mu}$, and l and C'_T depends on the same parameters as C and T'.

Proof of Lemma 4.21 . The first inequality of the lemma is a direct application of the estimate in Theorem 4.17. Indeed, $\forall x \in H, \forall T > 0$,

$$\begin{aligned} |w_T(0,x)| &= |u_T(0,x) - \lambda T - v(x)| \\ &= |Y_0^{T,x} - Y_0^x - \lambda T| \\ &\leq C(1 + |x|^{1+\mu}). \end{aligned} \tag{4.16}$$

Now, let us establish the gradient estimate. The process $(w_T(s, X_s^{t,x}))_{t \leq s \leq T}$ satisfies the following equation $\forall t \leq s \leq T$:

$$\begin{aligned} w_T(s, X_s^{t,x}) = w_T(T, X_T^{t,x}) &+ \int_s^T (f(X_r^{t,x}, Z_r^{T,t,x}) - f(X_r^{t,x}, Z_r^{t,x}))\mathrm{d}r \\ &- \int_s^T (Z_r^{T,t,x} - Z_r^{t,x})\mathrm{d}W_r. \end{aligned}$$

Now note that $\forall t \leq T$ and $t \leq s \leq T' \leq T$ we have

$$\begin{aligned} w_T(s, X_s^{t,x}) &= w_T(T', X_{T'}^{t,x}) - \int_s^{T'} (Z_r^{T,t,x} - Z_r^{t,x})\mathrm{d}W_r \\ &\quad + \int_s^{T'} (f(X_r^{t,x}, Z_r^{T,t,x}) - f(X_r^{t,x}, Z_r^{t,x}))\mathrm{d}r \\ &= w_{T-T'}(0, X_{T'}^{t,x}) - \int_t^{T'} (Z_r^{T,t,x} - Z_r^{t,x})\mathrm{d}W_r \\ &\quad + \int_s^{T'} (f(X_r^{t,x}, Z_r^{T,t,x} - Z_r^{t,x} + Z_r^{t,x}) - f(X_r^{t,x}, Z_r^{t,x}))\mathrm{d}r, \end{aligned}$$

where we have used equality (4.15) for the second equality.

We also recall that (see [32] Theorem 4.2 and [26] Theorem 3.8), $\forall x \in H, \forall s \in [t, T[$,

$$Z_s^{T,t,x} = \nabla_x u_T(s, X_s^{t,x})G \quad \text{and} \quad Z_s^{t,x} = \nabla_x v(X_s^{t,x})G.$$

Then we easily obtain that

$$Z_r^{T,t,x} - Z_r^{t,x} = \nabla_x w_T(r, X_r^{t,x})G.$$

Thus, applying the Bismut-Elworthy formula (see [32], Theorem 4.2), we get $\forall x, h \in H$, $\forall t < T$,

$$\nabla_x w_T(t,x)h = \mathbb{E}\int_t^{T'} [f\left(X_s^{t,x}, \nabla_x w_T(s, X_s^{t,x})G + Z_s^{t,x}\right) - f\left(X_s^{t,x}, Z_s^{t,x}\right)]U^h(s,t,x)\mathrm{d}s$$
$$+ \mathbb{E}\left[[w_{T-T'}(0, X_{T'}^{t,x})]U^h(T',t,x)\right],$$

where, $\forall 0 \le s \le T$, $\forall x \in H$,

$$U^h(s,t,x) = \frac{1}{s-t}\int_t^s \langle G^{-1}\nabla_x X_u^{t,x} h, \mathrm{d}W_u\rangle.$$

Let us recall that

$$\nabla_x X_s^{t,x} h = e^{(s-t)A}h;$$

then,

$$\mathbb{E}|U^h(s,t,x)|^2 = \frac{1}{|s-t|^2}\int_t^s |G^{-1}\nabla_x X_u^{t,x} h|^2 \mathrm{d}u \le \frac{C|h|^2}{s-t},$$

where C is independent of t, s, and x.

Thus we get $\forall x, h \in H$, $\forall t < T$, using inequality (4.16),

$$|\nabla_x w_T(t,x)h| \le C\int_t^{T'} \frac{\sqrt{\mathbb{E}(|\nabla_x w_T(s, X_s^{t,x})|^2)}|h|}{\sqrt{s-t}}\mathrm{d}s + C\frac{(1+|x|^{1+\mu})|h|}{\sqrt{T'-t}}.$$

We define

$$\varphi(t) = \sup_{x\in H}\frac{|\nabla_x w_T(t,x)|}{(1+|x|^{1+\mu})},$$

and we remark that $\varphi(t)$ is well defined $\forall t < T$. Indeed $\nabla_x w_T(t,x) = \nabla_x u_T(t,x) - \nabla_x v(x)$ and we have $|\nabla_x u_T(t,x)| \le C_T(T-t)^{-1/2}(1+|x|^{\mu})$ (see Theorem 4.2 in [33]) and $|\nabla_x v(x)| \le C(1+|x|^{1+\mu})$ (by Lemma 4.12). Then we obtain

$$|\nabla_x w_T(t,x)h| \le C\int_t^{T'} \frac{\varphi(s)}{\sqrt{s-t}}\sqrt{\mathbb{E}((1+|X_s^{t,x}|^{1+\mu})^2)}|h|\mathrm{d}s + C\frac{(1+|x|^{1+\mu})|h|}{\sqrt{T'-t}},$$

which leads to

$$\frac{|\nabla_x w_T(t,x)h|}{(1+|x|^{1+\mu})} \le C|h|\left(\int_t^{T'} \frac{\varphi(s)}{\sqrt{s-t}}\mathrm{d}s + \frac{1}{\sqrt{T'-t}}\right).$$

Taking the supremum over h such that $|h| = 1$ and $x \in H$, we have

$$\varphi(t) \le C\int_t^{T'} \frac{\varphi(s)}{\sqrt{s-t}}\mathrm{d}s + \frac{C}{\sqrt{T'-t}}.$$

Now note that we can rewrite the above inequality as follows:

$$\varphi(T'-t) \le C\int_0^t \frac{\varphi(T'-s)}{\sqrt{t-s}}\mathrm{d}s + \frac{C}{\sqrt{t}};$$

then by Lemma 7.1.1 in [41] we get

$$\varphi(T'-t) \le \frac{C}{\sqrt{t}} + C\theta\int_0^t E'(\theta(t-s))\frac{1}{\sqrt{s}}\mathrm{d}s,$$

where

$$\theta = (C\Gamma(1/2))^2 \quad \text{and} \quad E(z) = \sum_0^\infty \frac{z^{n/2}}{\Gamma(n/2+1)}.$$

Therefore, taking $t = T'$ leads us to

$$\varphi(0) \le \frac{C}{\sqrt{T'}} + C\theta\int_0^{T'} E'(\theta(T'-s))\frac{1}{\sqrt{s}}\mathrm{d}s,$$

which implies

$$|\nabla_x w_T(0,x)| \le C_{T'}(1+|x|^{1+\mu})\frac{1+\sqrt{T'}}{\sqrt{T'}}.$$

For the third inequality of Lemma 4.21, we have in the same way as for (4.14), $\forall x \in H$, $\forall T > 0$,

$$\begin{aligned} w_T(0,x) &= \mathbb{E}^{\mathbb{Q}^T}\left(g(X_T^x) - v(X_T^x)\right) \\ &= \mathscr{P}_T[g-v](x), \end{aligned}$$

where $\mathscr{P}_t$ is the Kolmogorov semigroup associated to the SDE

$$\mathrm{d}U_t^x = [AU_t^x + G\beta^T(t,U_t^x)]\mathrm{d}t + G\mathrm{d}W_t, \quad U_0^x = x, \quad t \ge 0,$$

and where $\beta^T(t,x) =$

$$\begin{cases} \frac{(f(x,\nabla u_T(t,x)G)-f(x,\nabla v(x)G))(\nabla u_T(t,x)G-\nabla v(x)G)^*}{|(\nabla u_T(t,x)-\nabla v(x))G|^2}\mathbb{1}_{t<T} & \\ \quad +\frac{(f(x,\nabla u_T(T,x)G)-f(x,\nabla v(x)G))(\nabla u_T(T,x)G-\nabla v(x)G)^*}{|(\nabla u_T(T,x)-\nabla v(x))G|^2}\mathbb{1}_{t\ge T} & \text{if } \nabla u_T(t,x) - \nabla v(x) \ne 0, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, $\forall x \in H$, $\forall T > 0$ we can write

$$|w_T(0,x) - w_T(0,y)| = |\mathscr{P}_T[g-v](x) - \mathscr{P}_T[g-v](y)|.$$

Then, as β^T is uniformly bounded in t and x, by Lemma 4.7, and thanks to Remark 4.6, we obtain, since $(g(\cdot) - v(\cdot))$ has polynomial growth of order $1+\mu$,

$$|w_T(0,x) - w_T(0,y)| \le C(1+|x|^{2+\mu} + |y|^{2+\mu})e^{-\hat{\eta}T}, \tag{4.17}$$

which concludes the proof of the lemma. $\square$

Now, let us come back to the proof of the theorem. The first estimate of Lemma 4.21 allows us to construct, by a diagonal procedure, a sequence $(T_i)_i \nearrow +\infty$ such that for a function $w : D \to \mathbb{R}$ defined on a countable dense subset D of H, the following holds:

$$\forall x \in D, \lim_{i\to+\infty} w_{T_i}(0,x) = w(x).$$

Then we fix $T' > 0$ and, by the second estimate of Lemma 4.21, we obtain that for every $x, y \in H$, for every $T \geq T'$,

$$|w_T(0,x) - w_T(0,y)| \leq \frac{C_{T'}}{\sqrt{T'}}(1 + |x|^{1+\mu} + |y|^{1+\mu})|x - y|.$$

By using this last inequality it is possible to extend w to the whole H. Indeed, if $x \notin D$, then there exists $(x_p)_{p\in\mathbb{N}} \in D^{\mathbb{N}}$ such that $x_p \to x$. Thus if we set $w(x) := \lim_{p\to+\infty} w(x_p)$, it is easy to check that $w_T(0,x) \underset{T\to+\infty}{\longrightarrow} w(x)$ for any $x \in H$.

Now, let us show that $w : H \to \mathbb{R}$ is a constant function. We have, by the third inequality of Lemma 4.21, $\forall x, y \in H$ and $T > 0$,

$$|w_T(0,x) - w_T(0,y)| \leq C(1 + |x|^{2+\mu} + |y|^{2+\mu})e^{-\hat{\eta}T}.$$

Applying the previous inequality with $T = T_i$ and taking the limit in i shows us that $x \mapsto w(x)$ is a constant function, namely, there exists $L_1 \in \mathbb{R}$ (independent of x) such that $\forall x \in H$,

$$\lim_i w_{T_i}(0,x) = L_1.$$

We remark that for any compact subset K of H, $\{w_T(0,\cdot)_{|K}; T > 1\}$ is a relatively compact subspace of the space of continuous functions $K \to \mathbb{R}$ for the uniform distance (denoted by $(\mathscr{C}(K,\mathbb{R}), ||\cdot||_{K,\infty})$ thanks to the two first inequalities of Lemma 4.21. Note now that L_1 is an accumulation point of $\{w_T(0,\cdot)_{|K}; T > 1\}$ since $w_{T_i}(\cdot)$ converges uniformly toward L_1 on any compact subset of H by the second inequality of Lemma 4.21.

Therefore, if we show that for every compact subset K of H, $\{w_T(0,\cdot)_{|K}; T > 1\}$ admits only one accumulation point (independently of K), it will imply that for all K compact subsets of H, $\forall x \in K$

$$\lim_{T\to+\infty} w_T(0,x) = L_1,$$

or, in other words, $\forall x \in H$,

$$\lim_{T\to+\infty} w_T(0,x) = L_1.$$

Now we claim that the accumulation point is unique. Let us assume that there exists another subsequence $(T'_i)_{i\in\mathbb{N}} \nearrow +\infty$ and $w_{\infty,K}(\cdot) \in \mathscr{C}(K,\mathbb{R})$ such that

$$\| w_{T'_i}(0,\cdot) - w_{\infty,K}(\cdot) \|_{K,\infty} \underset{i\to+\infty}{\longrightarrow} 0.$$

Then, by the third inequality of Lemma 4.21, there exists $L_{2,K}$ such that $\forall x \in K$, $w_{\infty,K}(x) = L_{2,K}$.

Let us write, $\forall x \in H$, $\forall T, S > 0$,

$$\begin{aligned}
w_{T+S}(0,x) &= Y_0^{T+S,x} - \lambda(T+S) - Y_0^x \\
&= Y_S^{T+S,x} - \lambda T - Y_S^x + \int_0^S (f(X_r^x, Z_r^{T+S,x}) - f(X_r^x, Z_r^x))\mathrm{d}r \\
&\qquad - \int_0^S (Z_r^{T+S,x} - Z_r^x)\mathrm{d}W_r \\
&= Y_S^{T+S,x} - \lambda T - Y_S^x + \int_0^S (Z_r^{T+S,x} - Z_r^x)\mathrm{d}\widetilde{W}_r^{T,S}
\end{aligned}$$

with

$$\widetilde{W}_t^{T,S} = -\int_0^t \beta^{T,S}(s, X_s^x)\mathrm{d}s + W_t \quad \forall t \in [0, S]$$

and where $\beta^{T,S}(t,x) =$

$$\begin{cases} \frac{(f(x,\nabla u_{T+S}(t,x)G)-f(x,\nabla v(x)G))((\nabla u_{T+S}(t,x)-\nabla v(x))G)^*}{|(\nabla u_{T+S}(t,x)-\nabla v(x))G|^2}\mathbb{1}_{t\le S}, & \\ +\frac{(f(x,\nabla u_{T+S}(S,x)G)-f(x,\nabla v(x)G))((\nabla u_{T+S}(S,x)-\nabla v(x))G)^*}{|(\nabla u_{T+S}(S,x)-\nabla v(x))G|^2}\mathbb{1}_{t>S} & \text{if } \nabla u_{T+S}(t,x)-\nabla v(x)\neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Taking the expectation with respect to the probability $\mathbb{Q}^{T,S}$ under which $W^{T,S}$ is a Brownian motion we get (using equality (4.15) for the third equality)

$$\begin{aligned} w_{T+S}(0,x) &= \mathbb{E}^{\mathbb{Q}^{T,S}}(Y_S^{T+S,x} - \lambda T - Y_S^x) \\ &= \mathbb{E}^{\mathbb{Q}^{T,S}}(w_{T+S}(S, X_S^x)) \\ &= \mathbb{E}^{\mathbb{Q}^{T,S}}(w_T(0, X_S^x)) \\ &= \mathscr{P}_S[w_T(0,\cdot)](x), \end{aligned} \tag{4.18}$$

where $\mathscr{P}_t$ is the Kolmogorov semigroup of the following SDE defined $\forall t \in \mathbb{R}_+$:

$$\mathrm{d}U_t^x = [AU_t^x + G\beta^{T,S}(t, U_t^x)]\mathrm{d}t + G\mathrm{d}W_t, \quad U_0^x = x.$$

This implies, substituting T by T_i' and S by $T_i - T_i'$ (up to a subsequence for $(T_i)_{i\in\mathbb{N}}$ such that $T_i > T_i'$), $\forall x \in H$,

$$w_{T_i}(0,x) = \mathscr{P}_{T_i - T_i'}[w_{T_i'}(0,\cdot)](x).$$

We recall that $\lim_i w_{T_i}(0,x) = L_1$ and we will show that the second term converges toward $L_{2,K}$ when $x \in K$. We have, $\forall x \in K$,

$$|\mathscr{P}_{T_i-T_i'}[w_{T_i'}(0,\cdot)](x) - L_{2,K}| \le |\mathscr{P}_{T_i-T_i'}[w_{T_i'}(0,\cdot)](x) - w_{T_i'}(0,x)| + |w_{T_i'}(0,x) - L_{2,K}|.$$

We recall that $|w_{T_i'}(0,x) - L_{2,K}| \underset{i\to+\infty}{\longrightarrow} 0$. Furthermore, if we denote by $U^{x,m,n}$ the mild solution of

$$\mathrm{d}U_t^{x,m,n} = [AU_t^{x,m,n} + G\beta_{m,n}^{T,S}(t, U_t^{x,m,n})]\mathrm{d}t + G\mathrm{d}W_t, \qquad U_0^{x,m,n} = x,$$

where $(\beta_{m,n}^{T,S})_{m\in\mathbb{N}^*, n\in\mathbb{N}^*}$ is the sequence of functions obtained by Lemma 4.7, then we have

$$\begin{aligned} |\mathscr{P}_{T_i-T_i'}[w_{T_i'}(0,\cdot)](x) - w_{T_i'}(0,x)| &= |\lim_n \lim_m \mathbb{E}(w_{T_i'}(0, U_{T_i-T_i'}^{x,m,n})) - w_{T_i'}(0,x)| \\ &\le \lim_n \lim_m |\mathbb{E}(w_{T_i'}(0, U_{T_i-T_i'}^{x,m,n})) - w_{T_i'}(0,x)| \\ &\le \lim_n \lim_m \mathbb{E}|w_{T_i'}(0, U_{T_i-T_i'}^{x,m,n}) - w_{T_i'}(0,x)| \\ &\le \lim_n \lim_m \mathbb{E}[C(1 + |U_{T_i-T_i'}^{x,m,n}|^{2+\mu} + |x|^{2+\mu})e^{-\hat{\eta}T_i'}] \\ &\le C(1+|x|^{2+\mu})e^{-\hat{\eta}T_i'}, \end{aligned}$$

where the third line is obtained thanks to the third inequality of Lemma 4.21. Therefore, letting $i \to +\infty$ shows us that $\forall x \in K$,

$$\mathscr{P}_{T_i-T_i'}[w_{T_i'}(0,\cdot)](x) \longrightarrow L_{2,K}.$$

Thus, for any compact subset K of H, $L_1 = L_{2,K}$, which, as mentioned before, implies that $\forall x \in H$,

$$\lim_{T\to+\infty} w_T(0,x) = L_1.$$

Finally we prove that this convergence holds with an explicit rate of convergence. Let us write, $\forall x \in H, \forall T > 0$,

$$\begin{aligned}|w_T(0,x) - L| &= \lim_{V\to+\infty} |w_T(0,x) - w_V(0,x)| \\ &= \lim_{V\to+\infty} |w_T(0,x) - \mathscr{P}_{V-T}[w_T(0,\cdot)](x)|\end{aligned}$$

thanks to equality (4.18), where $\mathscr{P}_t$ is the Kolmogorov semigroup associated to the following SDE defined $\forall t \in \mathbb{R}_+$:

$$\mathrm{d}U_t^x = [AU_t^x + \beta^{T,V-T}(t,U_t^x)]\mathrm{d}t + G\mathrm{d}W_t, \quad U_0 = x.$$

Now, if we denote by $U^{x,m,n}$ the mild solution of the SDE, $\forall t \geq 0$,

$$\mathrm{d}U_t^{x,m,n} = [AU_t^{x,m,n} + G\beta_{m,n}^{T,V-T}(t,U_t^x)]\mathrm{d}t + G\mathrm{d}W_t, \qquad U_0^{x,m,n} = x,$$

where $(\beta_{m,n}^{T,V-T})_{m\in\mathbb{N}^*,n\in\mathbb{N}^*}$ is the sequence of functions obtained by Lemma 4.7, then we have

$$\begin{aligned}|w_T(0,x) - L| &= \lim_{V\to+\infty} |w_T(0,x) - \lim_{n\to+\infty}\lim_{m\to+\infty} \mathbb{E}(w_T(0,U_{V-T}^{x,m,n}))| \\ &\leq \lim_{V\to+\infty}\lim_{n\to+\infty}\lim_{m\to+\infty} C\mathbb{E}(1+|x|^{2+\mu} + |U_{V-T}^{x,m,n}|^{2+\mu})e^{-\hat{\eta}T} \\ &\leq C(1+|x|^{2+\mu})e^{-\hat{\eta}T}\end{aligned}$$

thanks to the third estimate in Lemma 4.21. □

Remark 4.22. By the third inequality of Lemma 4.21, we have

$$|Y_0^{T,x} - Y_0^{T,0} - (Y_0^x - Y_0^0)| \leq C(1+|x|^{2+\mu})e^{-\hat{\eta}T}$$

i.e.,

$$|u(T,x) - u(T,0) - (v(x) - v(0))| \leq C(1+|x|^{2+\mu})e^{-\hat{\eta}T},$$

which provides possibly an efficient approximation for Y_0^x and $v(x)$.

4.5 Application to an ergodic control problem

In this section, we show how we can apply our results to an ergodic control problem. In this section we will still assume that Hypothesis 4.1 holds true and that F is Lipschitz and bounded and belongs to the class $\mathscr{G}^1$. Let U be a separable metric space. We define a control a as an $(\mathscr{F}_t)$-predictable U-valued process. We will assume the following.

Hypothesis 4.6. The functions $R : U \to H$, $L : H \times U \to \mathbb{R}$, and $g_0 : H \to \mathbb{R}$ are measurable and satisfy, for some constants $c > 0$, $C > 0$, and μ, the following:

1. $|R(a)| \leq c \;\; \forall a \in U$;
2. $L(\cdot, a)$ is continuous in x uniformly with respect to $a \in U$; furthermore $|L(x,a)| \leq C(1+|x|^{\mu})$ $\forall x \in H$ and $\forall a \in U$;
3. $g_0(\cdot)$ is continuous and $|g_0(x)| \leq C(1+|x|^{\mu})$ for all $x \in H$.

We denote by $(X_t^x)_{t\geq 0}$ the solution of (4.2). Given an arbitrary control a and $T>0$, we introduce the Girsanov density

$$\rho_T^{x,a} = \exp\left(\int_0^T G^{-1}R(a_s)\mathrm{d}W_s - \frac{1}{2}\int_0^T |G^{-1}R(a_s)|_\Xi^2 \mathrm{d}s\right)$$

and the probability $\mathbb{P}_T^a = \rho_T^a \mathbb{P}$ on $\mathscr{F}_T$. We introduce two costs. The first is the cost in the finite horizon:

$$J^T(x,a) := \mathbb{E}^{a,T}\int_0^T L(X_s^x, a_s)\mathrm{d}s + \mathbb{E}^{a,T} g_0(X_T^x),$$

where $\mathbb{E}^{a,T}$ denotes the expectation with respect to $\mathbb{P}_T^a$. The associated optimal control problem is to minimize the cost $J^T(x,a)$ over all controls $a^T : \Omega \times [0,T] \to U$, progressively measurable.

The second is called the ergodic cost and is the time averaged finite horizon cost:

$$J(x,a) := \limsup_{T\to+\infty} \frac{1}{T}\mathbb{E}^{a,T}\int_0^T L(X_s^x, a_s)\mathrm{d}s.$$

The associated optimal control problem is to minimize the cost $J(x,a)$ over all controls $a : \Omega \times [0,+\infty[\to U$, progressively measurable.

We notice that $W_t^a = W_t - \int_0^t G^{-1}R(a_s)\mathrm{d}s$ is a Wiener process on $[0,T]$ under $\mathbb{P}_T^a$ and that

$$\mathrm{d}X_t^x = (AX_t^x + F(X_t^x) + R(a_t))\mathrm{d}t + G\mathrm{d}W_t^a, \qquad t\in[0,T],$$

and this justifies our formulation of the control problem.

We want to show how our results can be applied to such an optimization problem to get an asymptotic expansion of the finite horizon cost involving the ergodic cost.

To apply our results, we first define the Hamiltonian in the usual way,

$$f_0(x,z) = \inf_{a\in U}\left\{L(x,a) + zG^{-1}R(a)\right\}, \tag{4.19}$$

and we remark that if $\forall x, z$, the infimum is atttained in (5.27), then by the Filippov theorem (see [59]), there exists a measurable function $\gamma : H\times\Xi^* \to U$ such that

$$f_0(x,z) = L(x,\gamma(x,z)) + zG^{-1}R(\gamma(x,z)).$$

Lemma 4.23. *Under the above assumptions, the Hamiltonian f_0 satisfies assumptions on f in Hypotheses 4.2, 4.3, 4.4 or 4.5.*

Proof. See Lemma 5.2 in [33]. □

We recall the following results about finite horizon cost

Lemma 4.24. *Assume that Hypotheses 4.1 and 4.6 hold true and that F is Lipschitz bounded and belong to the class $\mathscr{G}^1$, then for arbitrary control a,*

$$J^T(x,a) \geq u(T,x),$$

where $u(t,x)$ is the mild solution of

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = \mathscr{L}u(t,x) + f_0(x,\nabla u(t,x)G) & \forall (t,x)\in\mathbb{R}_+\times H,\\ u(0,x) = g_0(x) & \forall x\in H.\end{cases}$$

Furthermore, if $\forall x, z$ the infimum is attained in (5.27), then we have the equality

$$J^T(x,\overline{a}^T) = u(T,x),$$

where $\overline{a}_t^T = \gamma(X_t^{x,\overline{a}^T}, \nabla u(t, X_t^{x,\overline{a}^T})G)$.

Proof. See Theorem 5.3 in [32]. □

Similarly, for the ergodic cost we have the following result.

Lemma 4.25. *Assume that Hypotheses 4.1 and 4.6 hold true and that F is Lipschitz bounded and belongs to the class $\mathscr{G}^1$; then for arbitrary control a,*

$$J(x,a) \geq \lambda,$$

where (v,λ) is the mild solution of

$$\mathscr{L}v(x) + f_0(x,\nabla v(x)G) - \lambda = 0 \qquad \forall x \in H.$$

Furthermore, if $\forall x$, z the infimum is attained in (5.27), then we have the equality

$$J(x,\overline{a}) = \lambda,$$

where $\overline{a}_t = \gamma(X_t^{x,\overline{a}}, \nabla v(X_t^{x,\overline{a}})G)$.

Finally, we apply our result to obtain the following theorem.

Theorem 4.26. *Assume that Hypotheses 4.1 and 4.6 hold true and that F is Lipschitz bounded and belongs to the class $\mathscr{G}^1$. For any control a we have*

$$\liminf_{T\to+\infty} \frac{J^T(x,a)}{T} \geq \lambda.$$

Furthermore, if the infimum is attained in (5.27), then

$$J^T(x,\overline{a}^T) \underset{T\to+\infty}{\backsim} J(x,\overline{a})T + v(x) + L.$$

Proof. The proof is a straightforward consequence of the two previous lemmas above and of Theorem 4.20. □

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Chapitre 5

Comportement en temps long des solutions de viscosité d'EDP paraboliques semi-linéaires avec conditions de Neumann au bord

Résumé: Cet article est consacré à l'étude du comportement en temps long des solutions de viscosité d'EDP paraboliques avec conditions de Neumann au bord. Ce travail est la suite de [42] dans lequel une méthode probabiliste a été développée pour montrer que la solution d'une EDP parabolique semi-linéaire se comporte comme un terme linéaire λT shifté par une fonction v, où (v, λ) est la solution de l'EDP ergodique associée à l'EDP parabolique. Nous adaptons cette méthode en dimension finie par une méthode de pénalisation afin de pouvoir appliquer un résultat important de basic coupling estimate et avec une procédure de régularisation, afin de compenser le manque de régularité des coefficients en dimension finie. L'intérêt de notre méthode est qu'elle permet d'obtenir une vitesse de convergence explicite.
Mots clés: Équation différentielle stochastique rétrograde; équation différentielle stochastique rétrograde ergodique; équations de HJB; comportement en temps long; solutions de viscosité.

Abstract: This paper is devoted to the study of the large time behaviour of viscosity solutions of parabolic equations with Neumann boundary conditions. This work is the sequel of [42] in which a probabilistic method was developped to show that the solution of a parabolic semilinear PDE behaves like a linear term λT shifted with a function v, where (v, λ) is the solution of the ergodic PDE associated to the parabolic PDE. We adapt this method in finite dimension by a penalization method in order to be able to apply an important basic coupling estimate result and with the help of a regularization procedure in order to avoid the lack of regularity of the coefficients in finite dimension. The advantage of our method is that it gives an explicit rate of convergence.
Keywords:: Backward stochastic differential equations; ergodic backward stochastic differential equations; HJB equations; large time behaviour; viscosity solutions.

AMS classification: 35B40, 35K10, 60H30, 93E20.

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5.1 Introduction

We are concerned with the large time behaviour of solutions of the Cauchy problem with Neumann boundary conditions:

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = \mathscr{L}u(t,x) + f(x, \nabla u(t,x)\sigma), & \forall (t,x) \in \mathbb{R}_+ \times G, \\ \frac{\partial u(t,x)}{\partial n} + g(x) = 0, & \forall (t,x) \in \mathbb{R}_+ \times \partial G, \\ u(0,x) = h(x), & \forall x \in \overline{G}, \end{cases} \tag{5.1}$$

where, at least formally, $\forall \psi : \overline{G} \to \mathbb{R}$,

$$(\mathscr{L}\psi)(x) = \frac{1}{2}\operatorname{Tr}(\sigma^t\sigma\nabla^2\psi(x)) + \langle b(x), \nabla\psi(x)\rangle,$$

and $G = \{\phi > 0\}$ is a bounded convex open set of $\mathbb{R}^d$ with regular boundary. $u : \mathbb{R}_+ \times \overline{G} \to \mathbb{R}$ is the unknown function. We will assume that b is Lipschitz and σ is invertible. h is continuous and $g \in \mathscr{C}^1_{\text{lip}}(\overline{G})$. Furthermore we will assume that the non-linear term $f(x,z) : \mathbb{R}^d \times \mathbb{R}^{1\times d} \to \mathbb{R}$ is continuous in the first variable for all z and there exists $C > 0$ such that for all $x \in \mathbb{R}^d$, $\forall z_1, z_2 \in \mathbb{R}^{1\times d}$, $|f(x,z_1) - f(x,z_2)| \leq C|z_1 - z_2|$. Finally in order to obtain uniqueness for viscosity solutions of (5.1), we assume that ∂G is $W^{3,\infty}$ and that there exists $m \in \mathscr{C}((0,+\infty),\mathbb{R})$, $m(0^+) = 0$ such that $\forall x, y \in \overline{G}, \forall z \in \mathbb{R}^{1\times d}$,

$$|f(x,z) - f(y,z)| \leq m\left((1+|z|)|x-y|\right).$$

A lot of papers deal with the large time behaviour of parabolic PDEs (see for e.g. [61], [36], [44], [35] or [43]), but there are not a lot of them which deal with Neumann boundary conditions. In [8], Benachour and Dabuleanu study the large time behaviour of the Cauchy problem with zero Neumann boundary condition

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = \Delta u(t,x) + a|\nabla u(t,x)|^p, & \forall (t,x) \in \mathbb{R}_+^* \times G, \\ \frac{\partial u(t,x)}{\partial n} = 0, & \forall (t,x) \in \mathbb{R}_+ \times \partial G, \\ u(0,x) = h(x), & \forall x \in \overline{G}, \end{cases} \tag{5.2}$$

where $a \in \mathbb{R}$, $a \neq 0$, $p > 0$ and G is a bounded open set with smooth boundary of $\mathscr{C}^3$ class. The large time behaviour depends on the exponent p. If $p \in (0,1)$, and if h is a periodic function, then the solution is constant from a finite time. That is, there exist $T^* > 0$ and $c \in \mathbb{R}$ such that $u(t,x) = c$, for all $t > T^*$. When $p \geq 1$, any solution of (5.2) converges uniformly to a constant, as $t \to +\infty$.

In [45], Ishii establishes a result about the large time behaviour of a parabolic PDE in a bounded set with an Hamiltonian of first order $H(x,p)$, convex and coercive in p and with Neumann boundary coniditons.

In [4], Barles and Da Lio give a result for the large time behaviour of (5.1). Moreover, the result about the large time behaviour has been improved by Da Lio in [21] under the same hypotheses. In this last paper, the author studies the large time behaviour of non linear parabolic equation with Neumann boundary conditions on a smooth bounded domain $\overline{G}$:

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} + F(x, \nabla u(t,x), \nabla^2 u(t,x)) = \lambda, & \forall (t,x) \in \mathbb{R}_+ \times G, \\ L(x, \nabla u(t,x)) = \mu, & \forall (t,x) \in \mathbb{R}_+ \times \overline{G}, \\ u(0,x) = h(x), & \forall x \in G. \end{cases} \tag{5.3}$$

The spirit of this paper is slightly different from our work. Indeed, the result says that $\forall \lambda \in \mathbb{R}$, there exists $\mu \in \mathbb{R}$ such that (5.3) has a continuous viscosity solution. Moreover there exists a unique $\widetilde{\lambda}$ such that $\mu(\widetilde{\lambda}) = \widetilde{\lambda}$ for which the solution of (5.3) remains uniformly bounded in time $\widetilde{u}$. Then, there exists $\widetilde{u}_\infty$ solution of the ergodic PDE associated to (5.3) such that

$$\widetilde{u}(t,x) \underset{t\to+\infty}{\longrightarrow} \widetilde{u}_\infty(x), \qquad \text{uniformly in } \overline{G}.$$

Let us now state our main idea and result. Our method is purely probabilistic, which can be described as follows. First, let us consider $(X_t^x, K_t^x)_{t\geq 0}$ the solution of the following reflected SDE with values in $\overline{G}\times\mathbb{R}_+$,

$$\begin{cases} X_t^x = x + \int_0^t b(X_s^x)\mathrm{d}s + \int_0^t \nabla\phi(X_s^x)\mathrm{d}K_s^x + \int_0^t \sigma \mathrm{d}W_s, \quad t\geq 0,\\ K_t^x = \int_0^t \mathbb{1}_{\{X_s^x\in\partial G\}}\mathrm{d}K_s^x, \quad \forall t\geq 0, \end{cases}$$

where W is an $\mathbb{R}^d$-valued standard Brownian motion. Let (v,λ) be the solution of the following ergodic PDE,

$$\begin{cases} \mathscr{L}v(x) + f(x,\nabla v(x)\sigma) - \lambda = 0, \quad \forall x\in G,\\ \frac{\partial v(t,x)}{\partial n} + g(x) = 0, \quad \forall x\in\partial G. \end{cases}$$

Let $(Y^{T,x}, Z^{T,x})$ be the solution of the BSDE:

$$\begin{cases} \mathrm{d}Y_s^{T,x} = -f(X_s^x, Z_s^{T,x})\mathrm{d}s - g(X_s^x)\mathrm{d}K_s^x + Z_s^{T,x}\mathrm{d}W_s,\\ Y_T^{T,x} = h(X_T^x), \end{cases}$$

and (Y^x, Z^x, λ) be solution of the EBSDE:

$$\mathrm{d}Y_s^x = -(f(X_s^x, Z_s^x) - \lambda)\mathrm{d}s - g(X_s^x)\mathrm{d}K_s^x + Z_s^x\mathrm{d}W_s.$$

Then we have the following probabilistic representation:

$$\begin{cases} Y_s^{T,x} = u(T-s, X_s^x),\\ Y_s^x = v(X_s^x). \end{cases}$$

Then, in order to apply the method exposed in [42], we penalize and regularize the reflected process in order to apply the basic coupling estimates. Then, the use of a stability argument for BSDE helps us to conclude. Finally, we deduce that there exists a constant $L\in\mathbb{R}$ such that for all $x\in\mathbb{R}^d$,

$$Y_0^{T,x} - \lambda T - Y_0^x \underset{T\to+\infty}{\longrightarrow} L,$$

i.e.

$$u(T,x) - \lambda T - v(x) \underset{T\to+\infty}{\longrightarrow} L.$$

Our method uses not only purely probabilistic arguments, but also gives a rate of convergence:

$$|u(T,x) - \lambda T - v(x)| \leq Ce^{-\hat{\eta}T}.$$

The paper is organized as follows: In section 2, we introduce some notations. In section 3, we recall some existence and uniqueness results about a perturbed SDE, a reflected SDE, a BSDE and an EBSDE that will be useful for what follow in the paper. We recall how such BSDE and EBSDE are linked with PDE. In section 4, we study the large time behaviour of the solution of the BSDE taken at initial time when the horizon T of the BSDE increases. Then, we obtain a more precise result with an explicit rate of convergence in the Markovian case. In section 5, we apply our results to an optimal ergodic control problem.

5.2 Notations

We introduce some notations. Let E be an Euclidian space. We denote by $\langle \cdot, \cdot \rangle$ its scalar product and by $|\cdot|$ the associated norm. We denote by $B(x, M)$ the ball of center $x \in E$ and radius $M > 0$. Given $\phi \in B_b(E)$, the space of bounded and measurable functions $\phi : E \to \mathbb{R}$, we denote by $||\phi||_0 = \sup_{x \in E} |\phi(x)|$. If a function f is continuous and defined on a compact and convex subset $\overline{G}$ of $\mathbb{R}^d$, we define $f_{\mathbb{R}^d} := f(\Pi(x))$ where Π is the projection on $\overline{G}$. Note that $f_{\mathbb{R}^d}$ is continuous and bounded. $\mathscr{C}^k_{\text{lip}}(\overline{G})$ denotes the set of the functions of class $\mathscr{C}^k$ whose partial derivatives of order k are Lipschitz functions.

Given a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with a filtration $\mathscr{F}_t$, we consider the following classes of stochastic processes.

1. $L^p_{\mathscr{P}}(\Omega, \mathscr{C}([0,T]; E))$, $p \in [1, \infty)$, $T > 0$, is the space of predictable processes Y with continuous paths on $[0,T]$ such that

$$|Y|_{L^p_{\mathscr{P}}(\Omega, \mathscr{C}([0,T];E))} = \left(\mathbb{E} \sup_{t \in [0,T]} |Y_t|^p\right)^{1/p} < \infty.$$

2. $L^p_{\mathscr{P}}(\Omega, L^2([0,T]; E))$, $p \in [1, \infty)$, $T > 0$, is the space of predictable processes Y on $[0,T]$ such that

$$|Y|_{L^p_{\mathscr{P}}(\Omega, L^2([0,T],E))} = \left\{\mathbb{E}\left(\int_0^T |Y_t|^2 \mathrm{d}t\right)^{p/2}\right\}^{1/p} < \infty.$$

3. $L^2_{\mathscr{P},\text{loc}}(\Omega, L^2([0,\infty); E))$ is the space of predictable processes Y on $[0,\infty)$ which belong to the space $L^2_{\mathscr{P}}(\Omega, L^2([0,T]; E))$ for every $T > 0$. We define in the same way $L^p_{\mathscr{P},\text{loc}}(\Omega, \mathscr{C}([0,\infty); E))$.

In the sequel, we consider a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and a standard Brownian motion denoted by $(W_t)_{t \geq 0}$ with values in $\mathbb{R}^k$. $(\mathscr{F}_t)_{t \geq 0}$ will denote the natural filtration of W augmented with the family of $\mathbb{P}$-null sets of $\mathscr{F}$.

In this paper, C denotes a generic constant for which we specify the dependency on some parameters when it is necessary to do so. In this paper, we will consider only continuous viscosity solutions.

5.3 Preliminaries

5.3.1 The perturbed forward SDE

Let us consider the following stochastic differential equation with values in $\mathbb{R}^d$:

$$\begin{cases} \mathrm{d}X_t = d(X_t)\mathrm{d}t + b(t, X_t)\mathrm{d}t + \sigma \mathrm{d}W_t, \quad t \geq 0, \\ X_0 = x \in \mathbb{R}^d. \end{cases} \tag{5.4}$$

We will assume the following about the coefficients of the SDE:

Hypothesis 5.1. 1. $d : \mathbb{R}^d \to \mathbb{R}^d$ is locally Lipschitz, strict dissipative (i.e. there exists $\eta > 0$ such that for every $x, y \in \mathbb{R}^d$, $\langle d(x) - d(y), x - y \rangle \leq -\eta |x - y|^2$) and with polynomial growth (i.e. there exists $\mu > 0$ such that for every $x \in \mathbb{R}^d$, $|d(x)| \leq C(1 + |x|^\mu)$).

2. $b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ is bounded and measurable.

3. $\sigma \in \mathbb{R}^{d \times d}$ is invertible.

Definition 5.1. *We say that the SDE* (5.4) *admits a weak solution if there exists a new $\mathscr{F}$-Brownian motion $(\widehat{W}^x)_{t\geq 0}$ with respect to a new probability measure $\widehat{\mathbb{P}}$ (absolutely continuous with respect to $\mathbb{P}$), and an $\mathscr{F}$-adapted process $(\widehat{X}^x)_{t\geq 0}$ with continuous trajectories for which* (5.4) *holds with* $(W_t)_{t\geq 0}$ *replaced by* $(\widehat{W}_t^x)_{t\geq 0}$.

Lemma 5.2. *Assume that Hypothesis 5.1 holds true and that $b(t,\cdot)$ is Lipschitz uniformly w.r.t. $t \geq 0$. Then for every $x \in \mathbb{R}^d$, equation (5.4) admits a unique strong solution, that is, an adapted $\mathbb{R}^d$-valued process denoted by X^x with continuous paths satisfying $\mathbb{P}$-a.s.,*

$$X_t^x = x + \int_0^t d(X_s^x)\mathrm{d}s + \int_0^t b(s, X_s^x)\mathrm{d}s + \int_0^t \sigma \mathrm{d}W_s, \quad \forall t \geq 0.$$

Furthermore, we have the following estimate:

$$\mathbb{E}[|X_s^x|^p] \leq C(1 + |x|^p). \tag{5.5}$$

If b is only bounded and measurable then there exists a weak solution $(\widehat{X}, \widehat{W})$ and uniqueness in law holds. Furthermore, (5.5) *still holds (with respect to the new probability measure).*

Proof. For the first part of the lemma see [40], Theorem 3.3 in Chapter 1 or [56], Theorem 3.5. Estimates (5.5) is a simple consequence of Itô's formula. Weak existence and uniqueness in law are a direct consequence of a Girsanov's transformation. □

We define the Kolmogorov semigroup associated to Eq. (5.4) as follows: $\forall \phi : \mathbb{R}^d \to \mathbb{R}$ measurable with polynomial growth,

$$\mathscr{P}_t[\phi](x) = \mathbb{E}\phi(X_t^x).$$

Lemma 5.3 (Basic coupling estimate)**.** *Assume that Hypothesis 5.1 holds true and that $b(t,\cdot)$ is Lipschitz uniformly w.r.t. $t \geq 0$. Then there exists $\hat{c} > 0$ and $\hat{\eta} > 0$ such that for all $\phi : \mathbb{R}^d \to \mathbb{R}$ measurable and bounded (i.e. $\exists C, \mu > 0$ such that $\forall x \in \mathbb{R}^d$, $|\phi(x)| \leq C(1 + |x|^\mu)$),*

$$|\mathscr{P}_t[\phi](x) - \mathscr{P}_t[\phi](y)| \leq \hat{c}e^{-\hat{\eta}t}. \tag{5.6}$$

We stress the fact that $\hat{c}$ and $\hat{\eta}$ depend on b only through $\sup_{t\geq 0}\sup_{x\in\mathbb{R}^d}|b(t,x)|$.

Proof. See [55]. □

Corollary 5.4. *Relation (5.6) can be extended to the case in which b is only bounded and measurable and for all $t \geq 0$, there exists a uniformly bounded sequence of Lipschitz functions in x, $(b_n(t,\cdot))_{n\geq 1}$ (i.e. $\forall n \in \mathbb{N}$, $b_n(t,\cdot)$ is Lipschitz uniformly w.r.t. $t \geq 0$ and* $\sup_n \sup_t \sup_x |b_n(t,x)| < +\infty$*) such that*

$$\lim_n b_n(t,x) = b(t,x), \qquad \forall t \geq 0, \forall x \in R^d.$$

Clearly in this case in the definition of $\mathscr{P}_t[\phi]$ the mean value is taken with respect to the new probability measure $\widehat{\mathbb{P}}$.

Proof. It is enough to adapt the proof of Corollary 2.5 in [26]. The goal is to show that, if $\mathscr{P}^n$ denotes the Kolmogorov semigroup corresponding to equation (5.4) but with b replaced by b_n, then $\forall x \in \mathbb{R}^d$, $\forall t \geq 0$,

$$\mathscr{P}_t^n[\phi](x) \underset{n\to+\infty}{\longrightarrow} \mathscr{P}_t[\phi](x).$$

□

Remark 5.5. Similarly, if there exists a uniformly bounded sequence of Lipschitz functions $(b_{m,n}(t,\cdot))_{m\in\mathbb{N},n\in\mathbb{N}}$ (i.e. $\forall n \in \mathbb{N}, \forall m \in \mathbb{N}$, $F_{m,n}(t,\cdot)$ is Lipschitz uniformly w.r.t. $t \geq 0$ and $\sup_m \sup_n \sup_t \sup_x |b_{m,n}(t,x)| < +\infty$) such that

$$\lim_m \lim_n b_{m,n}(t,x) = b(t,x), \qquad \forall t \geq 0, \forall x \in \mathbb{R}^d,$$

then, if $\mathscr{P}^{m,n}$ is the Kolmogorov semigroup corresponding to equation (5.4) but with F replaced by $b_{m,n}$, we have $\forall t \geq 0$, $\forall x \in \mathbb{R}^d$,

$$\lim_m \lim_n \mathscr{P}_t^{m,n}[\phi](x) = \mathscr{P}[\phi](x),$$

which shows that relation (5.6) still holds.

We will need to apply the lemma above to some functions with particular form.

Lemma 5.6. *Let $f : \mathbb{R}^d \times \mathbb{R}^{1\times d} \to \mathbb{R}$ be continuous in the first variable and Lipschitz in the second one and ζ, ζ' be two continuous functions: $\mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{1\times d}$ be such that for all $s \geq 0$, $\zeta(s,\cdot)$ and $\zeta'(s,\cdot)$ are continuous. We define, for every $s \geq 0$ and $x \in \mathbb{R}^d$,*

$$\Upsilon(s,x) = \begin{cases} \frac{f(x,\zeta(s,x))-f(x,\zeta'(s,x))}{|\zeta(s,x)-\zeta'(s,x)|^2} {}^t(\zeta(s,x)-\zeta'(s,x)), & \text{if } \zeta(s,x) \neq \zeta'(s,x), \\ 0, & \text{if } \zeta(s,x) = \zeta'(s,x). \end{cases}$$

Then, there exists a uniformly bounded sequence of Lipschitz functions $(\Upsilon_{m,n}(s,\cdot))_{m\in\mathbb{N},n\in\mathbb{N}}$ (i.e., for every $m \in \mathbb{N}^*$ and $n \in \mathbb{N}^*$, $\Upsilon_n(s,\cdot)$ is Lipschitz and $\sup_m \sup_n \sup_s \sup_x |\Upsilon(s,x)| < +\infty$) such that for every $s \geq 0$ and for every $x \in \mathbb{R}^d$,

$$\forall x \in \mathbb{R}^d, \lim_{m\to+\infty} \lim_{n\to+\infty} \Upsilon_{m,n}(s,x) = \Upsilon(s,x).$$

Proof. See the proof of Lemma 3.5 in [26]. □

5.3.2 The reflected SDE

We consider a process X_t^x reflected in $\overline{G} = \{\phi > 0\}$. Let $(X_t^x, K_t^x)_{t\geq 0}$ denote the unknown of the following SDE:

$$\begin{cases} X_t^x = x + \int_0^t b(X_s^x)\mathrm{d}s + \int_0^t \nabla\phi(X_s^x)\mathrm{d}K_s^x + \int_0^t \sigma \mathrm{d}W_t, & t \in \mathbb{R}_+, \\ K_t^x = \int_0^t \mathbb{1}_{\{X_s^x \in \partial G\}}\mathrm{d}K_s^x. \end{cases} \tag{5.7}$$

Hypothesis 5.2. 1. $b : \overline{G} \to \mathbb{R}^d$ is Lipschitz.

2. $\sigma \in \mathbb{R}^{d\times d}$ is invertible.

We will make the following assumptions about G.

Hypothesis 5.3. 1. G is a bounded convex open set of $\mathbb{R}^d$.

2. $\phi \in \mathscr{C}^2_{\mathrm{lip}}(\mathbb{R}^d)$ and $G = \{\phi > 0\}$, $\partial G = \{\phi = 0\}$ and $\forall x \in \partial G$, $|\nabla\phi(x)| = 1$.

Remark 5.7. Let us denote by $\Pi(x)$ the projection of $x \in \mathbb{R}^d$ on $\overline{G}$. Let us extend the definition of b to $\mathbb{R}^d$ by setting, $\forall x \in \mathbb{R}^d$,

$$\widetilde{b}(x) := -x + (b(\Pi(x)) + \Pi(x)).$$

Note that $d(x) := -x$ is dissipative and that $p(x) := b(\Pi(x)) + \Pi(x)$ is Lipschitz and bounded. Therefore, $\widetilde{b}$ is weakly dissipative and satisfies Hypothesis 5.1.

Let us denote by $(X_t^{x,n})$ the solution of the following penalized SDE associated with (5.7):

$$X_t^{x,n} = x + \int_0^t \left[\widetilde{b}(X_s^{x,n}) + F_n(X_s^{x,n})\right] \mathrm{d}s + \int_0^t \sigma \mathrm{d}W_s,$$

where $\forall x \in \mathbb{R}^d$, $F_n(x) = -2n(x - \Pi(x))$.

Lemma 5.8. *Assume that the Hypotheses 5.2 and 5.3 hold true. Then for every $x \in \overline{G}$ there exists a unique pair of processes $(X_t^x, K_t^x)_{t\geq 0}$ with values in $(\overline{G}\times\mathbb{R}_+)$ and which belongs to the space $L^p_{\mathscr{P},loc}(\Omega, \mathscr{C}([0,+\infty[;\mathbb{R}^d)) \times L^p_{\mathscr{P},loc}(\Omega, \mathscr{C}([0,+\infty[;\mathbb{R}_+))$, $\forall p \in [1,+\infty[$, satisfying (5.7) and such that*

$$\eta_t^x := \int_0^t \nabla\phi(X_s^x)\mathrm{d}K_s^x, \qquad \textit{has bounded variation on } [0,T],\ \forall 0 \leq T < +\infty,\ \ \eta_0^x = 0,$$

and for all process z continuous and progressively measurable taking values in the closure $\overline{G}$ we have

$$\int_0^T (X_s^x - z_s)\mathrm{d}K_s^x \leq 0, \qquad \forall T \geq 0.$$

Finally, the following estimates holds for the convergence of the penalized process: for any $1 < q < p/2$, for any $T \geq 0$ there exists $C \geq 0$ such that

$$\mathbb{E} \sup_{0\leq t\leq T} |X_t^{x,n} - X_t^x|^p \leq \frac{C}{n^q}.$$

Proof. See Lemma 4.2 in [55]. □

5.3.3 The BSDE

Let us fix $T > 0$ and let us consider the following BSDE in finite horizon for an unknown process $(Y_s^{T,t,x}, Z_s^{T,t,x})_{s\in[t,T]}$ with values in $\mathbb{R}\times\mathbb{R}^{1\times d}$:

$$Y_s^{T,t,x} = \xi^T + \int_s^T f(X_r^{t,x}, Z_r^{T,t,x})\mathrm{d}r + \int_s^T g(X_r^{t,x})\mathrm{d}K_r^{t,x} - \int_s^T Z_r^{T,t,x}\mathrm{d}W_r, \qquad \forall s \in [t,T], \tag{5.8}$$

where $(X_s^{t,x}, K_s^{t,x})_{s\in[t,T]}$ is the solution of the SDE (5.7) starting from x at time t. If $t = 0$, we use the following standard notations $X_s^x = X_s^{0,x}$,$K_s^x = K_s^{0,x}$, $Y_s^{T,x} := Y_s^{T,0,x}$ and $Z_s^{T,x} = Z_s^{T,0,x}$. We will assume the following assumptions.

Hypothesis 5.4 (Path dependent case)**.** There exists $C > 0$, such that the function $f : \overline{G}\times\mathbb{R}^{1\times d} \to \mathbb{R}$ and ξ^T satisfy:

1. ξ^T is a real-valued random variable $\mathscr{F}_T$ measurable and $|\xi^T| \leq C$.
2. $\forall x \in \overline{G}$, $\forall z_1, z_2 \in \mathbb{R}^{1\times d}$, $|f(x,z_1) - f(x,z_2)| \leq C|z_1 - z_2|$.
3. $\forall z \in \mathbb{R}^{1\times d}$, $f(\cdot, z)$ is continuous.
4. $g \in \mathscr{C}^1_{\text{lip}}(\overline{G})$.

Lemma 5.9. *Assume that the Hypotheses 5.2, 5.3 and 5.4 hold true, then there exists a unique solution $(Y_s^{T,t,x}, Z_s^{T,t,x}) \in L^2_{\mathscr{P}}(\Omega, \mathscr{C}([0,T];\mathbb{R})) \times L^2_{\mathscr{P}}(\Omega, L^2([0,T];\mathbb{R}^{1\times d}))$.*

Proof. See Theorem 1.7 in [69]. □

Hypothesis 5.5 (Markovian case)**.** There exists $C > 0$ such that

1. $\xi^T = h(X_T^x)$, where $h : \overline{G} \to \mathbb{R}$ is continuous.
2. $\forall x \in \overline{G}$, $\forall z, z' \in \mathbb{R}^{1\times d}$, $|f(x,z) - f(x,z')| \leq C|z - z'|$.
3. $\forall z \in \mathbb{R}^{1\times d}$, $f(\cdot, z)$ is continuous.
4. $g \in \mathscr{C}^1_{\text{lip}}(\overline{G})$.

Let us consider the following semilinear PDE:

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} + \mathscr{L}u(t,x) + f(x, \nabla u(t,x)\sigma) = 0, & \forall (t,x) \in [0,T] \times G, \\ \frac{\partial u(t,x)}{\partial n} + g(x) = 0, & \forall (t,x) \in [0,T] \times \partial G, \\ u(T,x) = h(x), & \forall x \in G, \end{cases} \tag{5.9}$$

where $\mathscr{L}u(t,x) = \frac{1}{2}\operatorname{Tr}(\sigma^t\sigma\nabla^2 u(t,x)) + \langle b(x), \nabla u(t,x)\rangle$.

Lemma 5.10 (Existence)**.** *Assume that the Hypotheses 5.2, 5.3 and 5.5 hold true, then there exists a continuous viscosity solution to the PDE (5.9) given by*

$$u_T(t,x) = Y_t^{T,t,x}.$$

Proof. In our framework, $u_T(t,x) \in \mathscr{C}([0,T] \times \overline{G}; \mathbb{R})$. Indeed, first as in the proof of Theorem 3.1 in [72], we deduce the existence of a function $v^1 : \overline{G} \to \mathbb{R}$ which belongs to the space $\mathscr{C}^2_{\text{lip}}(\overline{G})$ and which is solution of Helmholtz's equation for some $\alpha \in \mathbb{R}$,

$$\begin{cases} \Delta v^1(x) - \alpha v^1(x) = 0, \\ \frac{\partial v^1(x)}{\partial n} + g(x) = 0. \end{cases}$$

We set $Y_s^{1,t,x} = v^1(X_s^{t,x})$ and $Z_s^{1,t,x} = \nabla v^1(X_s^{t,x})\sigma$. These processes verify, $\forall s \in [t,T]$,

$$Y_s^{1,t,x} = v^1(X_T^{t,x}) + \int_s^T [-\mathscr{L}v^1(X_r^{t,x})]\mathrm{d}r + \int_s^T g(X_r^{t,x})\mathrm{d}K_r^{t,x} - \int_s^T Z_r^{1,t,x}\mathrm{d}W_r,$$

where

$$\mathscr{L}v^1(x) = \frac{1}{2}\operatorname{Tr}(\sigma^t\sigma\nabla^2 v^1(x)) + \langle b(x), \nabla v^1(x)\rangle.$$

Then, if we define

$$\begin{aligned} \widetilde{Y}_s^{T,t,x} &= Y_s^{T,t,x} - v^1(X_s^{t,x}), \\ \widetilde{Z}_s^{T,t,x} &= Z_s^{T,t,x} - \nabla v^1(X_s^{t,x})\sigma, \end{aligned}$$

$(\widetilde{Y}^{T,t,x}, \widetilde{Z}^{T,t,x})$ satisfies the BSDE, $\forall s \in [t,T]$:

$$\begin{aligned} \widetilde{Y}_s^{T,t,x} = (h - v^1)(X_T^{t,x}) + \int_s^T \Big[f(X_r^{t,x}, \widetilde{Z}_r^{T,t,x} + \nabla v^1(X_r^{t,x})\sigma) + \mathscr{L}v^1(X_r^{t,x})\Big]\,\mathrm{d}r \\ - \int_s^T \widetilde{Z}_r^{T,t,x}\mathrm{d}W_r, \end{aligned}$$

which shows, since $v^1 \in \mathscr{C}^2_{\text{lip}}(\overline{G})$, that $\left((t,x) \mapsto \widetilde{Y}_t^{T,t,x}\right)$ is continuous. To show that $u_T(t,x)$ is a viscosity solution of (5.9) see [69], Theorem 4.3. □

Uniqueness for solutions of (5.9) holds under additional assumptions in our framework.

Hypothesis 5.6. 1. ∂G is of class $W^{3,\infty}$.

2. $\exists m \in \mathscr{C}((0,+\infty),\mathbb{R})$, $m(0^+)=0$ such that $\forall x,y \in \overline{G}, \forall z \in \mathbb{R}^{1\times d}$,

$$|f(x,z)-f(y,z)| \leq m\left((1+|z|)|x-y|\right).$$

Lemma 5.11 (Uniqueness)**.** *Assume that the Hypotheses 5.2, 5.3, 5.5 and 5.6 hold true. Then, uniqueness holds for viscosity solutions of* (5.9)*.*

Proof. See Theorem II.1 in [2]. □

Remark 5.12. By the following change of time: $\widetilde{u}_T(t,x) := u_T(T-t,x)$, we remark that $\widetilde{u}_T(t,x)$ is the unique viscosity solution of (5.1). Now remark that $\widetilde{u}_T(T,x) = u_T(0,x) = Y_0^{T,0,x} = Y_0^{T,x}$, therefore the large time behaviour of $Y_0^{T,x}$ is the same as that of the solution of equation (5.1).

5.3.4 The EBSDE

In this section, we consider the following ergodic BSDE for an unknown process $(Y_t^x, Z_t^x, \lambda)_{t\geq 0}$ with values in $\mathbb{R}\times\mathbb{R}^{1\times d}\times\mathbb{R}$:

$$Y_t^x = Y_T^x + \int_t^T (f(X_s^x, Z_s^x)-\lambda)\mathrm{d}s + \int_t^T g(X_s^x)\mathrm{d}K_s^x - \int_t^T Z_s^x \mathrm{d}W_s, \qquad \forall 0\leq t\leq T<+\infty. \tag{5.10}$$

Hypothesis 5.7. There exists $C>0$ and $\mu>0$ such that,

1. $\forall x \in \overline{G}$, $\forall z,z' \in \mathbb{R}^{1\times d}$, $|f(x,z)-f(x,z')| \leq C|z-z'|$.
2. $\forall z \in \mathbb{R}^{1\times d}$, $f(\cdot,z)$ is continuous.
3. $g \in \mathscr{C}^1_{\mathrm{lip}}(\overline{G})$.

Without loss of generality, we assume that $0 \in \overline{G}$.

Lemma 5.13 (Existence when Neumann boundary conditions are null)**.** *Assume that $g\equiv 0$ and that the Hypotheses 5.2, 5.3 and 5.7 hold true. Then there exists a solution $(Y^x,Z^x,\lambda) \in L^2_{\mathscr{P},loc}(\Omega,\mathscr{C}([0,+\infty[;\mathbb{R}))\times L^2_{\mathscr{P},loc}(\Omega,L^2([0,+\infty[;\mathbb{R}^{1\times d}))\times\mathbb{R}$ to (5.10). Moreover there exist $v:\overline{G}\to\mathbb{R}$ and $\xi:\overline{G}\to\mathbb{R}^{1\times d}$ measurable such that for every $x,y\in\overline{G}$, for all $t\geq 0$,*

$$\begin{aligned}
&Y_t^x = v(X_t^x), Z_t^x = \xi(X_t^x),\\
&v(0)=0,\\
&|v(x)-v(y)| \leq C,\\
&|v(x)-v(y)| \leq C|x-y|.
\end{aligned}$$

Proof. First let us recall that by Remark 5.7, one can replace b by its extension $\widetilde{b}$ which is weakly dissipative. Therefore, replacing f by $f_{\mathbb{R}^d}$, we obtain, by Theorem 4.4 in [55] that there exists $v:\overline{G}\to\mathbb{R}$ and $\xi:\overline{G}\to\mathbb{R}^{1\times d}$ measurable such that for every $x,y\in\overline{G}$, for all $t\geq 0$,

$$\begin{aligned}
&Y_t^x = v(X_t^x), Z_t^x = \xi(X_t^x),\\
&v(0)=0,\\
&|v(x)-v(y)| \leq C(1+|x|^{1+\mu}+|y|^{1+\mu}),\\
&|v(x)-v(y)| \leq C(1+|x|^{1+\mu}+|y|^{1+\mu})|x-y|.
\end{aligned}$$

And the result follows by the boundedness of $\overline{G}$. □

Lemma 5.14 (Existence)**.** *Assume that the Hypotheses 5.2, 5.3 and 5.10 hold true. Then there exists a solution* $(Y^x, Z^x, \lambda) \in L^2_{\mathscr{P},loc}(\Omega, \mathscr{C}([0,+\infty[;\mathbb{R})) \times L^2_{\mathscr{P},loc}(\Omega, L^2([0,+\infty[;\mathbb{R}^{1\times d})) \times \mathbb{R}$ *to the EBSDE (5.10). Moreover there exists* $v : \overline{G} \to \mathbb{R}$ *such that for every* $x, y \in \overline{G}$, *for all* $t \geq 0$,

$$\begin{aligned} &Y_t^x = v(X_t^x), \\ &|v(x)| \leq C, \\ &|v(x) - v(y)| \leq C|x-y|. \end{aligned}$$

Proof. First as in the proof of Theorem 3.1 in [72], we deduce the existence of a function $v^1 : \overline{G} \to \mathbb{R}$ which belongs to the space $\mathscr{C}^2_{\mathrm{lip}}(\overline{G})$ and is solution of Helmholtz's equation for some $\alpha \in \mathbb{R}$,

$$\begin{cases} \Delta v^1(x) - \alpha v^1(x) = 0, \\ \frac{\partial v^1(x)}{\partial n} + g(x) = 0. \end{cases}$$

Then, if we define $(Y_t^1 := v^1(X_t^x), Z_t^1 := \nabla v^1(X_t^x)\sigma)$, (Y^1, Z^1) satisfies, for every $0 \leq t \leq T < +\infty$:

$$Y_t^1 = Y_T^1 + \int_t^T \left[-\mathscr{L} v^1(X_s^x)\right] \mathrm{d}s + \int_t^T g(X_s^x) \mathrm{d}K_s^x - \int_t^T Z_s^1 \mathrm{d}W_s, \tag{5.11}$$

where

$$(\mathscr{L} v^1)(x) = \frac{1}{2} \operatorname{Tr}(\sigma^t \sigma \nabla^2 v^1) + \langle \widetilde{b}(x), \nabla v^1 \rangle.$$

Now consider the following EBSDE:

$$Y_t^2 = Y_T^2 + \int_t^T [f^2(X_s^x, Z_s^2) - \lambda] \mathrm{d}s - \int_t^T Z_s^2 \mathrm{d}W_s, \qquad \forall 0 \leq t \leq T < +\infty, \tag{5.12}$$

with $f^2(x,z) := \mathscr{L} v^1(x) + f(x, z + \nabla v^1(x)\sigma)$. Since $\forall z \in \mathbb{R}^{1\times d}$, $f^2(\cdot, z)$ is continuous and since for every $x \in \overline{G}$, $f^2(x, \cdot)$ is Lipschitz, one can apply Lemma 5.13 to obtain the existence of a solution $(Y_t^2 = v^2(X_t^x), Z_t^2 = \xi^2(X_t^x))$ to EBSDE (5.12) such that v^2 is continuous. We set

$$\begin{aligned} Y_t^x &= Y_t^1 + Y_t^2 = v^1(X_t^x) + v^2(X_t^x), \\ Z_t^x &= Z_t^1 + Z_t^2 = \nabla v^1(X_t^x)\sigma + \xi^2(X_t^x). \end{aligned}$$

Then (Y^x, Z^x, λ) is a solution of the EBSDE (5.10). □

Theorem 5.15 (Uniqueness of λ)**.** *Assume that the Hypotheses 5.2, 5.3 and 5.10 hold true. If* (Y^1, Z^1, λ^1) *and* (Y^2, Z^2, λ^2) *denote two solutions of the EBSDE* (5.10) *in the class of solutions* (Y, Z, λ) *such that* $\forall t \geq 0, |Y_t| \leq C$, $\mathbb{P}$*-a.s. and* $Z \in L^2_{\mathscr{P},loc}(\Omega, L^2([0,\infty[;\mathbb{R}^{1\times d})$, *then*

$$\lambda^1 = \lambda^2.$$

Proof. See Theorem 4.6 in [31]. □

Let us now state our main result of this section.

Theorem 5.16 (Uniqueness of solutions (Y, Z, λ))**.** *Assume that the Hypotheses 5.2, 5.3 and 5.10 hold true. Uniqueness holds for solutions* (Y, Z, λ) *of the EBSDE* (5.10) *in the class of solutions such that there exists* $v : \overline{G} \to \mathbb{R}$ *continuous,* $Y_s = v(X_s^x)$ *with* $v(0) = 0$, *and* $Z \in L^2_{\mathscr{P},loc}(\Omega, L^2([0,\infty[;\mathbb{R}^{1\times d})$.

Proof. Let $(Y^1 = v^1(X^x), Z^1, \lambda^1)$ and $(Y^2 = v^2(X^x), Z^2, \lambda^2)$ denote two solutions. Then from Theorem 5.15, we deduce that $\lambda^1 = \lambda^2 =: \lambda$.

Now, let us denote by $v : \overline{G} \to \mathbb{R}$, $v \in \mathscr{C}^2_{\text{lip}}(\overline{G})$ and solution of Helmholtz's equation for some $\alpha \in \mathbb{R}$

$$\begin{cases} \Delta v(x) - \alpha v(x) = 0, \\ \frac{\partial v(x)}{\partial n} + g(x) = 0. \end{cases}$$

Then, if we define $(Y_t := v(X_t^x), Z_t := \nabla v(X_t^x)\sigma)$, (Y, Z) satisfies, for every $0 \leq t \leq T < +\infty$:

$$Y_t = Y_T + \int_t^T [-\mathscr{L}v(X_s^x)]\,\mathrm{d}s + \int_t^T g(X_s^x)\mathrm{d}K_s^x - \int_t^T Z_s \mathrm{d}W_s, \tag{5.13}$$

where

$$(\mathscr{L}v)(x) = \frac{1}{2}\operatorname{Tr}(\sigma^t\sigma\nabla^2 v) + \langle\widetilde{b}(x), \nabla v\rangle.$$

Therefore, $(\widehat{Y}_t^1 = Y_t^1 - v(X_t^x), \widehat{Z}_t^1 = Z_t^1 - {}^t\nabla v(X_t^x)\sigma)$ satisfies the BSDE, $\forall 0 \leq t \leq T < +\infty$,

$$\widehat{Y}_t^1 = \widehat{Y}_T^1 + \int_t^T \widehat{f}(X_s^x, \widehat{Z}_s^x)\mathrm{d}s - \int_t^T \widehat{Z}_s^1 \mathrm{d}W_s,$$

where $\forall x, z \in \mathbb{R}^d \times \mathbb{R}^{1\times d}$,

$$\widehat{f}(x,z) = f\left(x, z + {}^t\nabla v^1(x)\sigma\right) - \lambda + \mathscr{L}v(x).$$

Then, let $(\widehat{Y}^{1,T,t,x}, \widehat{Z}^{1,T,t,x})$ be the solution of the following BSDE, $\forall s \in [t,T]$,

$$\widehat{Y}_s^{1,T,t,x} = (v^1 - v)(X_T^{t,x}) + \int_t^T \widehat{f}(X_s^x, \widehat{Z}_s^{1,T,t,x})\mathrm{d}s - \int_t^T \widehat{Z}_s^{1,T,t,x}\mathrm{d}W_s.$$

By uniqueness of solutions to BSDE, we deduce that

$$v^1(x) - v(x) = \widehat{Y}_0^{1,T,0,x}.$$

Now, we fix infinitely differentiable functions $\rho_\varepsilon : \mathbb{R}^d \to \mathbb{R}^+$ bounded together with their derivatives of all order, such that: $\int_{\mathbb{R}^d} \rho_\varepsilon(x)\mathrm{d}x = 1$ and

$$\operatorname{supp}(\rho_\varepsilon) \subset \left\{\xi \in \mathbb{R}^d : |\xi| \leq \varepsilon\right\},$$

where supp denotes the support. Then we define $\forall n \in \mathbb{N}$,

$$(F_n)_\varepsilon(x) = \int_{\mathbb{R}^d} \rho_\varepsilon(y) F_n(x-y)\mathrm{d}y,$$
$$\widetilde{b}_\varepsilon(x) = \int_{\mathbb{R}^d} \rho_\varepsilon(y)\widetilde{b}(x-y)\mathrm{d}y.$$

Let us denote by $X^{t,x,n,\varepsilon}$ the solution of the following SDE, $\forall s \geq t$,

$$X_s^{t,x,n,\varepsilon} = x + \int_t^s \left(\widetilde{b}_\varepsilon + (F_n)_\varepsilon\right)(X_r^{t,x,n,\varepsilon})\mathrm{d}r + \int_t^s \sigma \mathrm{d}W_r,$$

and let $(Y^{1,T,t,x,n,\varepsilon}, Z^{1,T,t,x,n,\varepsilon})$ be the solution of the following BSDE, $\forall s \in [t,T]$,

$$Y_s^{1,T,t,x,n,\varepsilon} = (v^1 - v)(X_s^{t,x,n,\varepsilon}) + \int_t^s \widehat{f}(X_r^{t,x,n,\varepsilon}, Z_r^{1,T,t,x,n,\varepsilon})\mathrm{d}r - \int_s^T Z_r^{1,T,t,x,n,\varepsilon}\mathrm{d}W_s.$$

Then by a stability result, (see for e.g. Lemma 2.3 of [11]), we deduce that

$$\lim_{n\to+\infty}\lim_{\varepsilon\to+\infty} Y_0^{1,T,0,x,n,\varepsilon} = \widehat{Y}_0^{1,T,0,x} = v^1(x) - v(x). \tag{5.14}$$

Similarly, defining $(Y^{2,T,t,x}, Z^{2,T,t,x})$ and $(Y^{2,T,t,x,n,\varepsilon}, Z^{2,T,t,x,n,\varepsilon})$ in the same way, we deduce that

$$\lim_{n\to+\infty}\lim_{\varepsilon\to+\infty} Y_0^{2,T,0,x,n,\varepsilon} = \widehat{Y}_0^{2,T,0,x} = v^2(x) - v(x).$$

Furthermore, by Theorem 4.2 (or Theorem 4.2 in [32]), if we define $u^{1,T,n,\varepsilon}(t,x) := Y_t^{1,T,t,x,n,\varepsilon}$, then $(x \mapsto u^{1,T,n,\varepsilon}(t,x))$ is continuously differentiable for all $t \in [0,T[$, and $\forall s \in [t,T[$,

$$Z_s^{1,T,t,x,n,\varepsilon} = {}^t\nabla u^{1,T,n,\varepsilon}(s, X_s^{t,x,n,\varepsilon})\sigma.$$

Similarly, we define $u^{2,T,n,\varepsilon}(t,x) := Y_t^{2,T,t,x,n,\varepsilon}$ and then

$$Z_s^{2,T,t,x,n,\varepsilon} = {}^t\nabla u^{2,T,n,\varepsilon}(s, X_s^{t,x,n,\varepsilon})\sigma.$$

Therefore, taking $t = 0$, $\forall T > 0$,

$$\begin{aligned} u^{1,T,n,\varepsilon}(0,x) - u^{2,T,n,\varepsilon}(0,x) &= (v^1 - v^2)(X_T^{x,n,\varepsilon}) - \int_0^T (Z_s^{1,T,x,n,\varepsilon} - Z_s^{2,T,x,n,\varepsilon})\mathrm{d}W_s \\ &\quad + \int_0^T \left[\widehat{f}(X_s^{x,n,\varepsilon}, Z_s^{1,T,x,n,\varepsilon}) - \widehat{f}(X_s^{x,n,\varepsilon}, Z_s^{2,T,x,n,\varepsilon})\right]\mathrm{d}s \\ &= (v^1 - v^2)(X_T^{x,n,\varepsilon}) \\ &\quad - \int_0^T (Z_s^{1,T,x,n,\varepsilon} - Z_s^{2,T,x,n,\varepsilon})(-\beta(s, X_s^{x,n,\varepsilon})\mathrm{d}s + \mathrm{d}W_s), \end{aligned}$$

where

$$\beta^T(s,x) = \begin{cases} \frac{f(x,{}^t\nabla u^{1,T,n,\varepsilon}(x)\sigma) - f(x,{}^t\nabla u^{2,T,n,\varepsilon}(x)\sigma)}{|{}^t\nabla u^{1,T,n,\varepsilon}(x)\sigma - {}^t\nabla u^{2,T,n,\varepsilon}(x)\sigma|^2}\mathbb{1}_{t<T}, & \text{if } \nabla u^{1,T,n,\varepsilon}(t,x) \neq \nabla u^{2,T,n,\varepsilon}(t,x), \\ 0, & \text{otherwise.} \end{cases}$$

The process $(\beta^T(s, X_s^{x,n,\varepsilon}))_{s\in[0,T]}$ is progressively measurable and bounded, therefore, we can apply Girsanov's Theorem to obtain that there exists a new probability measure $\mathbb{Q}^T$ equivalent to $\mathbb{P}$ under which $(W_t - \int_0^t \beta(s, X_s^{x,n,\varepsilon})\mathrm{d}s)_{t\in[0,T]}$ is a Brownian motion. Therefore, denoting by $E^{\mathbb{Q}^T}$ the expectation with respect to the probability $\mathbb{Q}^T$,

$$\begin{aligned} u^{1,T,n,\varepsilon}(0,x) - u^{2,T,n,\varepsilon}(0,x) &= \mathbb{E}^{\mathbb{Q}^T}\left[(v^1 - v^2)(X_T^{x,n,\varepsilon})\right] \\ &= \mathscr{P}_T[v^1 - v^2](x), \end{aligned}$$

where $\mathscr{P}_t$ is the Kolmogorov semigroup associated to the following SDE, $\forall t \geq 0$,

$$U_t^x = x + \int_0^t \left(\widetilde{b}_\varepsilon + (F_n)_\varepsilon\right)(U_s^x)\mathrm{d}s + \int_0^t \sigma\beta(s, U_s^x)\mathrm{d}s + \int_0^s \sigma\mathrm{d}W_s.$$

By Corollary 5.4 and Remark 5.6, we deduce that

$$|u^{1,T,n,\varepsilon}(0,x) - u^{2,T,n,\varepsilon}(0,x) - (u^{1,T,n,\varepsilon}(0,0) - (u^{2,T,n,\varepsilon}(0,0))| \leq Ce^{-\hat{\eta}T}.$$

Therefore, thanks to (5.14),

$$|v^1(x) - v^2(x) - (v^1(0) - v^2(0))| \leq Ce^{-\hat{\eta}T}.$$

Therefore, since $v^1(0) = v^2(0) = 0$, letting $T \to +\infty$ we deduce that

$$v^1(x) = v^2(x), \forall x \in \overline{G}.$$

□

We recall the link of such EBSDE with ergodic PDE. Let us consider the following ergodic semilinear PDE for which the unknown is a pair (v, λ):

$$\begin{cases} \mathscr{L}v(x) + f(x, \nabla v(x)\sigma) - \lambda = 0, & \forall x \in G, \\ \frac{\partial v(x)}{\partial n} + g(x) = 0, & \forall x \in \partial G. \end{cases} \tag{5.15}$$

Lemma 5.17 (Existence of ergodic viscosity solutions). *Assume that the Hypotheses 5.2, 5.3 and 5.7 hold true then the solution (v, λ) of Lemma 5.14 is a viscosity solution of* (5.15).

Proof. Note that v is continuous by Lemma 5.14. The proof of this result is very classical and can be easily adapted from [69]. □

Lemma 5.18 (Uniqueness of ergodic viscosity solutions). *Assume that the Hypotheses 5.2, 5.3, 5.6 and 5.10 hold true. Then uniqueness holds for viscosity solutions (v, λ) of* (5.15) *in the class of (continuous) viscosity solutions such that $\exists a \in \mathbb{R}^d$, $v^1(a) = v^2(a)$.*

Proof. Let (v^1, λ^1) and (v^2, λ^2) be two continuous viscosity solutions of (5.15). First we show that $\lambda^1 = \lambda^2$. Let us fix $0 \le t < T < +\infty$, and let us consider $(Y^{1,T,t,x}, Z^{1,T,t,x})$ the solution of the following BSDE in finite horizon, $\forall s \in [t, T]$,

$$Y_s^{1,T,t,x} = v^1(X_T^{t,x}) + \int_s^T [f(X_r^{t,x}, Z_r^{1,T,t,x}) - \lambda^1]\mathrm{d}r + \int_s^T g(X_r^{t,x})\mathrm{d}K_r^{t,x} - \int_s^T Z_r^{1,T,t,x}\mathrm{d}W_r.$$

And we define $(Y^{2,T,t,x}, Z^{2,T,t,x})$ similarly, replacing λ^1 by λ^2. By Lemma 5.10, we deduce that $u^{1,T}(t,x) = Y_t^{1,T,t,x}$ is a viscosity solution of (5.9). Since v^1 is also a viscosity solution of (5.9), it follows from Lemma 5.11 that $\forall t \in [0, T], \forall x \in \overline{G}$,

$$u^{1,T}(t,x) = v^1(x).$$

Of course, similarly, $\forall t \in [0, T]$, $\forall x \in \overline{G}$,

$$u^{2,T}(t,x) = v^2(x).$$

Then, taking $t = 0$, $\forall T > 0$,

$$\begin{aligned} u^{1,T}(0,x) - u^{2,T}(0,x) &= v^1(X_T^x) - v^2(X_T^x) + \int_0^T [f(X_s^x, Z_s^{1,T,x}) - f(X_s^x, Z_s^{2,T,x})]\mathrm{d}s \\ &\quad + (\lambda^2 - \lambda^1)T - \int_0^T \left(Z_s^{1,T,x} - Z_s^{2,T,x}\right)\mathrm{d}W_s \\ &= v^1(X_T^x) - v^2(X_T^x) - \int_0^T (Z_s^{1,T,x} - Z_s^{2,T,x})(-\beta_s \mathrm{d}s + \mathrm{d}W_s), \end{aligned}$$

where, $\forall s \in [0, T]$,

$$\beta_s = \begin{cases} \frac{(f(X_s^x, Z_s^{1,T,x}) - f(X_s^x, Z_s^{2,T,x}))^t (Z_s^{1,T,x} - Z_s^{2,T,x})}{|Z_s^{1,T,x} - Z_s^{2,T,x}|^2}, & \text{if } Z_s^{1,T,x} \neq Z_s^{2,T,x}, \\ 0, & \text{otherwise.} \end{cases}$$

Since $(\beta_s)_{s\in[0,T]}$ is a measurable and bounded process, by Girsanov's theorem, there exists a new probability $\mathbb{Q}^T$ equivalent to $\mathbb{P}$ under which $(W_t - \int_0^t \beta_s \mathrm{d}s)_{t\in[0,T]}$ is a Brownian motion. Taking the expectation with respect to this new probability, we get

$$\frac{u^{1,T}(0,x) - u^{2,T}(0,x)}{T} = \frac{\mathbb{E}^{Q^T}(v^1(X_T^x) - v^2(X_T^x))}{T} + \lambda^2 - \lambda^1.$$

Since v^1 and v^2 are continuous and therefore bounded on $\overline{G}$, letting $T \to +\infty$ we deduce that

$$\lambda^1 = \lambda^2.$$

Applying the same argument as that in Theorem 5.16, we deduce the uniqueness. □

5.4 Large time behaviour

5.4.1 First behaviour

We recall that $(Y_s^{T,x}, Z_s^{T,x})_{s\geq 0}$ denotes the solution of the finite horizon BSDE (5.8) with $t = 0$ and that $(Y_s^x, Z_s^x, \lambda)_{s\geq 0}$ denotes the solution of the EBSDE (5.10).

Theorem 5.19. *Assume that the Hypotheses 5.2, 5.3 and 5.4 hold true (path dependent case), then, $\forall x \in \overline{G}$, $\forall T > 0$:*

$$\left|\frac{Y_0^{T,x}}{T} - \lambda\right| \leq \frac{C}{T}.$$

In particular,

$$\frac{Y_0^{T,x}}{T} \underset{T\to+\infty}{\longrightarrow} \lambda,$$

uniformly in $\overline{G}$.
Assume that the Hypotheses 5.2, 5.3, 5.5 and 5.6 hold true (Markovian case). Then, $\forall x \in \overline{G}$, $\forall T > 0$:

$$\left|\frac{Y_0^{T,x}}{T} - \lambda\right| \leq \frac{C}{T}.$$

i.e.

$$\left|\frac{u(T,x)}{T} - \lambda\right| \leq \frac{C}{T},$$

where u is the viscosity solution of (5.1). *In particular,*

$$\frac{u(T,x)}{T} = \frac{Y_0^{T,x}}{T} \underset{T\to+\infty}{\longrightarrow} \lambda,$$

uniformly in $\overline{G}$.

Proof. The proof is identical to the proof of Theorem 4.1 in [42]. Note that the proof is even simpler since we work with a bounded subset G of $\mathbb{R}^d$ and then for any probability $\mathbb{Q}^T$, $\mathbb{E}^{\mathbb{Q}^T}[\sup_{0\leq t\leq T}|X_t|^\mu] \leq C$, where C depends only on G and μ. Note that the proof gives an important result

$$|u_T(0,x) - \lambda T - v(x)| \leq C, \tag{5.16}$$

which will be useful for what follow. Finally note that for the Markovian case, Hypothesis 5.6 is added in order to obtain uniqueness of viscosity solutions of (5.1). $\square$

5.4.2 Second and third behaviour

In this section we introduce a new set of hypothesis without loss of generality. Note that it is the same as Hypothesis 5.5 but with modified assumptions for b. However we write it again for reader's convenience. The remark immediately following this new set of hypothesis justifies the fact that there is no loss of generality. Let us denote by $(Y_s^{t,x}, Z_s^{t,x}, \lambda)_{s\geq 0}$ the solution of the EBSDE (5.10) when X^x is replaced by $X^{t,x}$. We recall that this solution satisfies

$$Z_s^{t,x} = \nabla v^1(X_s^{t,x})\sigma + Z_s^2. \tag{5.17}$$

Hypothesis 5.8. There exists $C > 0$ such that

1. $b : \mathbb{R}^d \to \mathbb{R}^d$ is $\mathscr{C}^1$ Lipschitz and dissipative (i.e. $\exists \eta > 0$ such that $\forall x, y \in \mathbb{R}^d$, $\langle b(x) - b(y), x - y\rangle \leq -\eta|x-y|^2$).
2. $\xi^T = h(X_T^x)$, where $h : \overline{G} \to \mathbb{R}$ is continuous.
3. $\forall x \in \overline{G}$, $\forall z, z' \in \mathbb{R}^{1\times k}$, $|f(x,z) - f(x,z')| \leq C|z - z'|$.
4. $\forall z \in \mathbb{R}^{1\times k}$, $f(\cdot, z)$ is continuous.
5. $g \in \mathscr{C}^1_{\text{lip}}(\overline{G})$.

Remark 5.20. Note that assuming b to be $\mathscr{C}^1$ Lipschitz and dissipative is not restrictive. Indeed, let us consider $b : \overline{G} \to \mathbb{R}^d$ only Lipschitz. Let us recall that the purpose of this paper is to study the large time behaviour of the viscosity solution of

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = \mathscr{L}u(t,x) + f(x, \nabla u(t,x)G), & \forall (t,x) \in \mathbb{R}_+ \times \overline{G}, \\ \frac{\partial u(t,x)}{\partial n} + g(x) = 0, & \forall (t,x) \in \mathbb{R}_+ \times \partial G, \\ u(0,x) = h(x), & \forall x \in G. \end{cases}$$

Now, we define, $\forall x \in \mathbb{R}^d$, $\widetilde{b}(x) := -x + (b(\Pi(x)) + \Pi(x))$. Note that $\widetilde{b}$ is equal to b on $\overline{G}$. Furthermore,

$$\langle \widetilde{b}(x), \nabla u(t,x)\rangle + f(x, \nabla u(t,x)\sigma) = \langle -x, \nabla u(t,x)\rangle + \widetilde{f}(x, \nabla u(t,x)\sigma),$$

where $\widetilde{f}(x,z) = f(x,z) + \langle b(\Pi(x)) + \Pi(x), z\sigma^{-1}\rangle$ is a continuous function in x and Lipschitz in z. Therefore, under our assumptions, we can always consider the case b being $\mathscr{C}^1$ Lipschitz and dissipative by replacing b by $(x \mapsto -x)$ and f by $\widetilde{f}$ if necessary.

Theorem 5.21. *Assume that the Hypotheses 5.2, 5.3, 5.6 and 5.8 hold true. Then there exists $L \in \mathbb{R}$ such that,*

$$\forall x \in \overline{G},\ Y_0^{T,x} - \lambda T - Y_0^x \underset{T\to+\infty}{\longrightarrow} L,$$

i.e.

$$\forall x \in \overline{G},\ u(T,x) - \lambda T - v(x) \underset{T\to+\infty}{\longrightarrow} L,$$

where u is the viscosity solution of (5.1) *and v is the viscosity solution of* (5.15). *Furthermore the following rate of convergence holds*

$$|Y_0^{T,x} - \lambda T - Y_0^x - L| \leq Ce^{-\hat{\eta}T},$$

i.e.

$$|u_T(0,x) - \lambda T - v(x) - L| \leq Ce^{-\hat{\eta}T}.$$

Proof. Let us start by defining

$$\begin{aligned} &u_T(t,x) := Y_t^{T,t,x} \\ &w_T(t,x) := u_T(t,x) - \lambda(T-t) - v(x). \end{aligned}$$

We recall that $Y_s^{T,t,x} = u_T(s, X_s^{t,x})$ and that $Y_s^x = v(X_s^x)$.

Note that $(x \mapsto w_T(0,x))$ is continuous and bounded uniformly in T by (5.16). Therefore one can extend the definition of $w_T(0,x)$ to the whole $\mathbb{R}^d$ into a continuous and uniformly bounded in T function by setting $w_{T,\mathbb{R}^d}(0,x) := w_T(0,\Pi(x))$ where Π is the projection on $\overline{G}$.

We recall that for all $T, S \geq 0$, u_T is the unique solution of

$$\begin{cases} \frac{\partial u_T(t,x)}{\partial t} + \mathscr{L} u_T(t,x) + f(x, \nabla u_T(t,x) G) = 0, & \forall (t,x) \in [0,T] \times G, \\ \frac{\partial u_T(t,x)}{\partial n} + g(x) = 0, & \forall (t,x) \in [0,T] \times \partial G, \\ u_T(T,x) = h(x), & \forall x \in G, \end{cases}$$

and that u_{T+S} is the unique solution of

$$\begin{cases} \frac{\partial u_{T+S}(t,x)}{\partial t} + \mathscr{L} u_{T+S}(t,x) + f(x, \nabla u_{T+S}(t,x)\sigma) = 0, & \forall (t,x) \in [0,T+S] \times G, \\ \frac{\partial u_{T+S}(t,x)}{\partial n} + g(x) = 0, & \forall (t,x) \in [0,T+S] \times \partial G, \\ u_{T+S}(T+S,x) = h(x), & \forall x \in G. \end{cases}$$

By uniqueness of viscosity solutions, it implies that $u_T(0,x) = u_{T+S}(S,x)$, for all $x \in \overline{G}$, and then,

$$w_T(0,x) = w_{T+S}(S,x), \forall x \in \overline{G}. \tag{5.18}$$

For every $T \geq t$, the process $(w_T(s, X_s^{t,x}))_{s\in[t,T]}$ satisfies the following BSDE in infinite horizon, $\forall t \leq s \leq T < +\infty$,

$$\begin{aligned} w_T(s, X_s^{t,x}) &= w_T(T, X_T^{t,x}) + \int_s^T [f(X_r^{t,x}, Z_r^{T,t,x}) - f(X_r^{t,x}, Z_r^{t,x})]\mathrm{d}r \\ &\qquad - \int_s^T (Z_r^{T,t,x} - Z_r^{t,x})\mathrm{d}W_r \\ &= h(X_T^{t,x}) - v(X_T^{t,x}) + \int_s^T [f(X_r^{t,x}, Z_r^{T,t,x}) - f(X_r^{t,x}, Z_r^{t,x})]\mathrm{d}r \\ &\qquad - \int_s^T (Z_r^{T,t,x} - Z_r^{t,x})\mathrm{d}W_r. \end{aligned} \tag{5.19}$$

Since we do not have a basic coupling estimate Lemma for the reflected process $X^{t,x}$, we will use an approximation procedure. We fix infinitely differentiable functions $\rho_\varepsilon : \mathbb{R}^d \to \mathbb{R}^+$ bounded together with their derivatives of all order, such that: $\int_{\mathbb{R}^d} \rho_\varepsilon(x)\mathrm{d}x = 1$ and

$$\mathrm{supp}(\rho_\varepsilon) \subset \left\{ \xi \in \mathbb{R}^d : |\xi| \leq \varepsilon \right\}$$

where supp denotes the support. Then we define $\forall n \in \mathbb{N}$,

$$(F_n)_\varepsilon(x) = \int_{\mathbb{R}^d} \rho_\varepsilon(y) F_n(x-y)\mathrm{d}y.$$

It is well known that $(F_n)_\varepsilon$ is $\mathscr{C}^\infty$. Furthermore, $(F_n)_\varepsilon$ is still 0-dissipative. Let $(X_s^{t,x,n,\varepsilon})_{s\geq t}$ be the solution of

$$X_s^{t,x,n,\varepsilon} = x + \int_t^s (b + (F_n)_\varepsilon)(X_r^{t,x,n,\varepsilon})\mathrm{d}r + \int_t^s \sigma \mathrm{d}W_r, \quad \forall s \geq t,$$

and $(Y_s^{2,t,x,\alpha,n,\varepsilon}, Z_s^{2,t,x,\alpha,n,\varepsilon})_{s\geq t}$ be the solution of the following monotonic BSDE in infinite horizon, $\forall t \leq s \leq T < +\infty$,

$$\begin{aligned} Y_s^{2,t,x,\alpha,n,\varepsilon} = Y_T^{2,t,x,\alpha,n,\varepsilon} &+ \int_s^T \left[f(X_r^{t,x,n,\varepsilon}, Z_r^{2,t,x,\alpha,n,\varepsilon}) - \alpha Y_r^{2,t,x,\alpha,n,\varepsilon} \right] \mathrm{d}r \\ &- \int_s^T Z_r^{2,t,x,\alpha,n,\varepsilon} \mathrm{d}W_r. \end{aligned}$$

By Theorem 4.4 in [55], there exist sequences $\varepsilon_m \underset{m\to+\infty}{\longrightarrow} 0$, $\beta(n) \underset{n\to+\infty}{\longrightarrow} +\infty$ and $\alpha_k \underset{k\to+\infty}{\longrightarrow} 0$ such that for all $T \geq t$,

$$\lim_{k\to+\infty}\lim_{n\to+\infty}\lim_{m\to+\infty}\mathbb{E}\int_t^T \left|Z_s^{2,t,x,\alpha_k,\beta(n),\varepsilon_m} - Z_s^2\right|^2 \mathrm{d}s = 0. \tag{5.20}$$

In what follows, we will use the following notation. If $q^{\alpha,n,\varepsilon}$ denotes a function depending on the parameters α, n and ε, then

$$\lim_{\alpha,n,\varepsilon} q^{\alpha,n,\varepsilon} := \lim_{k\to+\infty}\lim_{n\to+\infty}\lim_{m\to+\infty} q^{\alpha_k,\beta(n),\varepsilon_m}.$$

Now, if we define, for all $s \geq t$,

$$\widetilde{Z}_s^{t,x,\alpha,n,\varepsilon} := (\nabla v^1)_{\mathbb{R}^d}(X_s^{t,x,n,\varepsilon})\sigma + Z_s^{2,t,x,\alpha,n,\varepsilon},$$

by the dominated convergence theorem and thanks to (5.17) and (5.20), for all $T \geq t$

$$\lim_{\alpha,n,\varepsilon}\mathbb{E}\int_t^T |\widetilde{Z}_s^{t,x,\alpha,n,\varepsilon} - Z_s^{t,x}|^2 \mathrm{d}s = 0. \tag{5.21}$$

Note that by Theorem 4.2 in [54], if we define $v^{2,\alpha,n,\varepsilon}(x) := Y_0^{x,\alpha,n,\varepsilon}$, then $v^{2,\alpha,n,\varepsilon}$ is $\mathscr{C}^1$ and $\forall s \geq t$,

$$Z_s^{2,t,x,\alpha,n,\varepsilon} = {}^t\nabla v^{2,\alpha,n,\varepsilon}(X_s^{t,x,n,\varepsilon})\sigma.$$

Therefore, we have the following representation, $\forall s \geq t$,

$$\begin{aligned}\widetilde{Z}_s^{t,x,\alpha,n,\varepsilon} &= {}^t\nabla(v^1)_{\mathbb{R}^d}(X_s^{t,x,n,\varepsilon})\sigma + {}^t\nabla v^{2,\alpha,n,\varepsilon}(X_s^{t,x,n,\varepsilon})\sigma \\ &=: {}^t\nabla\widetilde{v}^{\alpha,n,\varepsilon}(X_s^{t,x,n,\varepsilon})\sigma.\end{aligned} \tag{5.22}$$

Let us denote by $(\overline{Y}_s^{T,t,x,\alpha,n,\varepsilon}, \overline{Z}_s^{T,t,x,\alpha,n,\varepsilon})_{s\geq t}$ the solution of the following BSDE in finite horizon, $\forall s \in [t,T]$,

$$\begin{aligned}\overline{Y}_s^{T,t,x,\alpha,n,\varepsilon} &= w_{T,\mathbb{R}^d}(T, X_T^{t,x,n,\varepsilon}) - \int_s^T \overline{Z}_r^{T,t,x,\alpha,n,\varepsilon}\mathrm{d}W_r \\ &\quad + \int_s^T \left[f_{\mathbb{R}^d}(X_r^{t,x,n,\varepsilon}, \overline{Z}_r^{T,t,x,\alpha,n,\varepsilon} + \widetilde{Z}_r^{t,x,\alpha,n,\varepsilon}) - f_{\mathbb{R}^d}(X_r^{t,x,n,\varepsilon}, \widetilde{Z}_r^{t,x,\alpha,n,\varepsilon})\right]\mathrm{d}r \\ &= (h-v)_{\mathbb{R}^d}(X_T^{t,x,n,\varepsilon}) + \int_s^T \widetilde{f}^{\alpha,n,\varepsilon}(s, \overline{Z}_r^{T,t,x,\alpha,n,\varepsilon})\mathrm{d}r - \int_s^T \overline{Z}_r^{T,t,x,\alpha,n,\varepsilon}\mathrm{d}W_r,\end{aligned} \tag{5.23}$$

where, for all $r \geq t$, $z \in \mathbb{R}^{1\times d}$,

$$\widetilde{f}^{\alpha,n,\varepsilon}(r,z) := f_{\mathbb{R}^d}(X_r^{t,x,n,\varepsilon}, z + \widetilde{Z}_r^{t,x,\alpha,n,\varepsilon}) - f_{\mathbb{R}^d}(X_r^{t,x,n,\varepsilon}, \widetilde{Z}_r^{t,x,\alpha,n,\varepsilon}).$$

We define, for all $z \in \mathbb{R}^{1\times d}$,

$$\widetilde{f}(r,z) := f_{\mathbb{R}^d}(X_r^{t,x}, z + Z_r^{t,x}) - f_{\mathbb{R}^d}(X_r^{t,x}, Z_r^{t,x}).$$

The assumption (A2) of [11] is satisfied, indeed:

$$\forall z_1, z_2 \in \mathbb{R}^{1\times d}, |\widetilde{f}^{\alpha,n,\varepsilon}(s,z_1) - \widetilde{f}^{\alpha,n,\varepsilon}(s,z_2)| \leq C|z_1 - z_2|,$$

$$\mathbb{E}\left[\int_t^T |\widetilde{f}^{\alpha,n,\varepsilon}(s,0)|^2\mathrm{d}s\right] + \sup_{t\leq s\leq T}\mathbb{E}\left[|X_s^{t,x,n,\varepsilon}|^2\right] \leq C.$$

Now we show that the assumption (A3) of [11] is satisfied. We have, $\forall s \in [t,T]$,

$$\begin{aligned}
&\mathbb{E}\left[\int_s^T |\widetilde{f}^{\alpha,n,\varepsilon}(r, Z_r^{T,t,x} - Z_r^{t,x}) - \widetilde{f}(r, Z_r^{T,t,x} - Z_r^{t,x})|^2 \mathrm{d}r\right] \\
&= \mathbb{E}\left[\int_s^T |f_{\mathbb{R}^d}(X_r^{t,x,n,\varepsilon}, Z_r^{T,t,x} - Z_r^{t,x} + \widetilde{Z}_r^{t,x,\alpha,n,\varepsilon}) - f_{\mathbb{R}^d}(X_r^{t,x,n,\varepsilon}, \widetilde{Z}_r^{t,x,\alpha,n,\varepsilon}) \right.\\
&\qquad \left. - f_{\mathbb{R}^d}(X_r^{t,x}, Z_r^{T,t,x}) + f_{\mathbb{R}^d}(X_r^{t,x}, Z_r^{t,x})|^2 \mathrm{d}r\right] \\
&\leq C\mathbb{E}\left[\int_s^T |f_{\mathbb{R}^d}(X_r^{t,x,n,\varepsilon}, Z_r^{T,t,x} - Z_r^{t,x} + \widetilde{Z}_r^{t,x,\alpha,n,\varepsilon}) - f_{\mathbb{R}^d}(X_r^{t,x,n,\varepsilon}, Z_r^{T,t,x})|^2 \mathrm{d}r\right] \\
&\quad + C\mathbb{E}\left[\int_s^T |f_{\mathbb{R}^d}(X_r^{t,x,n,\varepsilon}, Z_r^{T,t,x}) - f_{\mathbb{R}^d}(X_r^{t,x}, Z_r^{T,t,x})|^2 \mathrm{d}r\right] \\
&\quad + C\mathbb{E}\left[\int_s^T |f_{\mathbb{R}^d}(X_r^{t,x,n,\varepsilon}, \widetilde{Z}_r^{t,x,\alpha,n,\varepsilon}) - f_{\mathbb{R}^d}(X_r^{t,x,n,\varepsilon}, Z_r^{t,x})|^2 \mathrm{d}r\right] \\
&\quad + C\mathbb{E}\left[\int_s^T |f_{\mathbb{R}^d}(X_r^{t,x,n,\varepsilon}, Z_r^{t,x}) - f_{\mathbb{R}^d}(X_r^{t,x}, Z_r^{t,x})|^2 \mathrm{d}r\right] \\
&\leq C\mathbb{E}\left[\int_s^T |\widetilde{Z}_r^{t,x,\alpha,n,\varepsilon} - Z_r^{t,x}|^2 \mathrm{d}r\right] + C\mathbb{E}\left[\int_s^T |f_{\mathbb{R}^d}(X_r^{t,x,n,\varepsilon}, Z_r^{T,t,x}) - f_{\mathbb{R}^d}(X_r^{t,x}, Z_r^{T,t,x})|^2 \mathrm{d}r\right] \\
&\quad + C\mathbb{E}\left[\int_s^T |f_{\mathbb{R}^d}(X_r^{t,x,n,\varepsilon}, Z_r^{t,x}) - f_{\mathbb{R}^d}(X_r^{t,x}, Z_r^{t,x})|^2 \mathrm{d}r\right].
\end{aligned}$$

Then (5.21) implies that the first term converges toward 0. Furthermore, since $\lim_{\varepsilon,n} \mathbb{E}\sup_{t\leq s\leq T} |X_s^{t,x,n,\varepsilon} - X_s^{t,x}|^2 = 0$, we have

$$|X_s^{t,x,n,\varepsilon} - X_s^{t,x}| \xrightarrow{\mathbb{P}\otimes \mathrm{d}t} 0, \text{ as } \varepsilon \to 0, n \to +\infty,$$

and

$$|f_{\mathbb{R}^d}(X_r^{t,x,n,\varepsilon}, Z_r^{T,t,x}) - f_{\mathbb{R}^d}(X_r^{t,x}, Z_r^{T,t,x})|^2 \leq C(1 + |Z_r^{T,t,x}|^2)$$

which shows the uniform integrability of $|f_{\mathbb{R}^d}(X_r^{t,x,n,\varepsilon}, Z_r^{T,t,x}) - f_{\mathbb{R}^d}(X_r^{t,x}, Z_r^{T,t,x})|^2$. Therefore, the second term converges toward 0. The same argument applied to the third term shows that this last term also converges toward 0.

Furthermore, by continuity and boundedness of $(h-v)_{\mathbb{R}^d}$, we deduce that:

$$\lim_{\alpha,n,\varepsilon} \mathbb{E}\left[|(h-v)_{\mathbb{R}^d}(X_T^{t,x,n,\varepsilon}) - (h-v)_{\mathbb{R}^d}(0, X_T^{t,x})|^2\right] = 0.$$

Thus assumption (A3) of [11] is satisfied. Therefore, by Lemma 2.3 of [11] applied to (5.23) and (5.19), we obtain:

$$\lim_{\alpha,n,\varepsilon} \overline{Y}_t^{T,t,x,\alpha,n,\varepsilon} = w_{T,\mathbb{R}^d}(t,x).$$

Thus, $\forall x \in \overline{G}$,

$$\lim_{\alpha,n,\varepsilon} \overline{Y}_t^{T,t,x,\alpha,n,\varepsilon} = w_T(t,x). \tag{5.24}$$

We define

$$\overline{w}_T^{\alpha,n,\varepsilon}(t,x) = \overline{Y}_t^{T,t,x,\alpha,n,\varepsilon}.$$

Similarly to equation (5.18), we deduce that, $\forall T, S \geq 0$

$$\overline{w}_T^{\alpha,n,\varepsilon}(0,x) = \overline{w}_{T+S}^{\alpha,n,\varepsilon}(S,x). \tag{5.25}$$

Now we are in force to apply the method exposed in [42] for the quantity $\overline{w}_T^{\alpha,n,\varepsilon}(0,x)$ with slight modifications.

First we establish the following Lemma.

Lemma 5.22. *Under the hypotheses of Theorem 5.21, $\exists C > 0$, $\forall x, y \in \overline{G}$, $\forall T > 0$, $\forall 0 < T' \leq T$, $\exists C_{T'}$,*

$$\begin{aligned}
&|\overline{w}_T^{\alpha,n,\varepsilon}(0,x)| \leq C,\\
&|\nabla_x \overline{w}_T^{\alpha,n,\varepsilon}(0,x)| \leq \frac{C_{T'}}{\sqrt{T'}},\\
&|\overline{w}_T^{\alpha,n,\varepsilon}(0,x) - \overline{w}_T^{\alpha,n,\varepsilon}(0,y)| \leq Ce^{-\hat{\eta}T}.
\end{aligned}$$

We stress the fact that C depends only on η, σ, $\overline{G}$. The constant $C_{T'}$ depends only on the same constant and T'.

Proof. The first estimate is a direct consequence of Girsanov's theorem. Indeed, we have,

$$\begin{aligned}
\overline{Y}_0^{T,x,\alpha,n,\varepsilon} &= (h-v)_{\mathbb{R}^d}(X_T^{x,n,\varepsilon}) - \int_0^T \overline{Z}_r^{T,x,\alpha,n,\varepsilon} \mathrm{d}W_r\\
&\quad + \int_0^T \left[f_{\mathbb{R}^d}(X_r^{x,n,\varepsilon}, \overline{Z}_r^{T,x,\alpha,n,\varepsilon} + \widetilde{Z}_r^{x,\alpha,n,\varepsilon}) - f_{\mathbb{R}^d}(X_r^{x,n,\varepsilon}, \widetilde{Z}_r^{x,\alpha,n,\varepsilon}) \right] \mathrm{d}r\\
&= w_T(0, X_T^{x,n,\varepsilon}) - \int_0^T \overline{Z}_r^{T,x,\alpha,n,\varepsilon}(-\beta_r \mathrm{d}r + \mathrm{d}W_r),
\end{aligned}$$

where

$$\beta_r := \begin{cases} \frac{(f_{\mathbb{R}^d}(X_r^{x,n,\varepsilon}, \overline{Z}_r^{T,x,\alpha,n,\varepsilon} + \widetilde{Z}_r^{x,\alpha,n,\varepsilon}) - f_{\mathbb{R}^d}(X_r^{x,n,\varepsilon}, \widetilde{Z}_r^{x,\alpha,n,\varepsilon}))^t(\overline{Z}_r^{T,x,\alpha,n,\varepsilon})}{|\overline{Z}_r^{T,x,\alpha,n,\varepsilon}|^2}, & \text{if } \overline{Z}_r^{T,x,\alpha,n,\varepsilon} \neq 0,\\ 0, & \text{otherwise.} \end{cases}$$

Since β is a measurable and bounded process, there exists a new probability equivalent to $\mathbb{P}$, $\mathbb{Q}^{T,\alpha,n,\varepsilon}$ under which $(W_s - \int_0^s \beta_r \mathrm{d}r)_{r \in [0,T]}$ is a Brownian motion. Therefore, thanks to estimate (5.16):

$$\begin{aligned}
|\overline{Y}_0^{T,x,\alpha,n,\varepsilon}| &\leq \mathbb{E}^{Q^{T,\alpha,n,\varepsilon}} |w_T(0, X_T^{x,n,\varepsilon})|\\
&\leq C.
\end{aligned}$$

Let us establish the second and third inequality of the lemma. First we notice that thanks to equation (5.18), $\forall 0 \leq T' < T$, $\forall s \in [t, T']$,

$$\begin{aligned}
\overline{Y}_s^{T,t,x,\alpha,n,\varepsilon} &= \overline{w}_{T,\mathbb{R}^d}^{\alpha,n,\varepsilon}(T', X_{T'}^{t,x,n,\varepsilon}) - \int_s^{T'} \overline{Z}_r^{T,t,x,\alpha,n,\varepsilon} \mathrm{d}W_r\\
&\quad + \int_s^{T'} \left[f_{\mathbb{R}^d}(X_r^{t,x,n,\varepsilon}, \overline{Z}_r^{T,t,x,\alpha,n,\varepsilon} + \widetilde{Z}_r^{t,x,\alpha,n,\varepsilon}) - f_{\mathbb{R}^d}(X_r^{t,x,n,\varepsilon}, \widetilde{Z}_r^{t,x,\alpha,n,\varepsilon}) \right] \mathrm{d}r\\
&= \overline{w}_{T-T',\mathbb{R}^d}^{\alpha,n,\varepsilon}(0, X_{T'}^{t,x,n,\varepsilon}) + \int_s^{T'} \widetilde{f}^{\alpha,n,\varepsilon}(s, \overline{Z}_r^{T,t,x,\alpha,n,\varepsilon}) \mathrm{d}r - \int_s^{T'} \overline{Z}_r^{T,t,x,\alpha,n,\varepsilon} \mathrm{d}W_r.
\end{aligned}$$

We recall that we have the following representation:

$$\widetilde{Z}_s^{t,x,\alpha,n,\varepsilon} = \nabla \widetilde{v}(X_s^{t,x,n,\varepsilon})\sigma.$$

Furthermore, by Theorem 4.2 (or Theorem 4.2 in [32]), as $\overline{w}_T^{\alpha,n,\varepsilon}(t,x) := \overline{Y}_t^{T,t,x,\alpha,n,\varepsilon}$, $(x \mapsto \overline{u}_T^{\alpha,n,\varepsilon}(t,x))$ is continuously differentiable for all $t \in [0,T[$ and $\forall s \in [t,T[$,

$$\overline{Z}_s^{t,x,\alpha,n,\varepsilon} = \nabla \overline{w}_T^{\alpha,n,\varepsilon}(s, X_s^{t,x,n,\varepsilon})\sigma.$$

Therefore, we can apply the same method as exposed in [42] to obtain the second and third estimate. □

Let us conclude the proof. From Lemma 5.22, we derive, by the same arguments as in [42] that there exists $L^{\alpha,n,\varepsilon} \in \mathbb{R}$ such that $\forall x \in \mathbb{R}^d$,

$$|\overline{w}_T^{\alpha,n,\varepsilon}(0,x) - L^{\alpha,n,\varepsilon}| \leq Ce^{-\hat{\eta}T}. \tag{5.26}$$

Therefore,

$$\begin{cases} \overline{w}_T^{\alpha,n,\varepsilon}(0,x) \leq Ce^{-\hat{\eta}T} + L^{\alpha,n,\varepsilon}, \\ L^{\alpha,n,\varepsilon} \leq Ce^{-\hat{\eta}T} + \overline{w}_T^{\alpha,n,\varepsilon}(0,x), \end{cases}$$

which implies by (5.24) that

$$\begin{cases} w_T(0,x) \leq Ce^{-\hat{\eta}T} + \liminf_{\alpha,n,\varepsilon} L^{\alpha,n,\varepsilon}, \\ \limsup_{\alpha,n,\varepsilon} L^{\alpha,n,\varepsilon} \leq Ce^{-\hat{\eta}T} + w_T(0,x). \end{cases}$$

Then,

$$\limsup_{\alpha,n,\varepsilon} L^{\alpha,n,\varepsilon} \leq 2Ce^{-\hat{\eta}T} + \liminf_{\alpha,n,\varepsilon} L^{\alpha,n,\varepsilon}.$$

Letting $T \to +\infty$ implies that there exists $L \in \mathbb{R}$ such that

$$\lim_{\alpha,n,\varepsilon} L^{\alpha,n,\varepsilon} = L.$$

Coming back to (5.26) and passing to the limit gives us the result:

$$|w_T(0,x) - L| \leq Ce^{-\hat{\eta}T}.$$

□

5.5 Application to an ergodic control problem

In this section, we show how we can apply our results to an ergodic control problem. We assume that Hypotheses 5.2 and 5.3 hold. Let U be a separable metric space. We define a control a as an $(\mathscr{F}_t)_{t\geq 0}$-predictable U-valued process. We will assume the following.

Hypothesis 5.9. The functions $R : U \to \overline{G}$, $L : \overline{G} \times U \to \mathbb{R}$ and $g_0 : \overline{G} \to \mathbb{R}$ are measurable and satisfy, for some $C > 0$,

1. $|R(a)| \leq C, \quad \forall a \in U$.
2. $L(\cdot, a)$ is continuous in x uniformly with respect to $a \in U$. Furthermore $|L(\cdot,a)| \leq C$, furthermore $|L(x,a)| \leq C, \quad \forall x \in \overline{G}, \forall x \in \overline{G}, \forall a \in U$.
3. $h_0(\cdot)$ is continuous.
4. $g \in \mathscr{C}^1(\overline{G})$.

We denote by $(X_t^x)_{t\geq 0}$ the solution of (5.7). Given an arbitrary control a and $T>0$, we introduce the Girsanov density

$$\rho_T^{x,a} = \exp\left(\int_0^T \sigma^{-1}R(a_s)\mathrm{d}W_s - \frac{1}{2}\int_0^T |\sigma^{-1}R(a_s)|^2\mathrm{d}s\right)$$

and the probability $\mathbb{P}_T^a = \rho_T^a\mathbb{P}$ on $\mathscr{F}_T$. We introduce two costs. The first one is the cost in finite horizon:

$$J^T(x,a) := \mathbb{E}^{a,T}\left[\int_0^T L(X_s^x,a_s)\mathrm{d}s + \int_0^T g(X_s^x)\mathrm{d}K_s^x\right] + \mathbb{E}^{a,T}h_0(X_T^x),$$

where $\mathbb{E}^{a,T}$ denotes the expectation with respect to $\mathbb{P}_T^a$. The associated optimal control problem is to minimize the cost $J^T(x,a)$ over all controls $a^T:\Omega\times[0,T]\to U$, progressively measurable. The second one is called the ergodic cost and is the time averaged finite horizon cost:

$$J(x,a) := \limsup_{T\to+\infty}\frac{1}{T}\mathbb{E}^{a,T}\left[\int_0^T L(X_s^x,a_s)\mathrm{d}s + \int_0^T g(X_s^x)\mathrm{d}K_s^x\right].$$

The associated optimal control problem is to minimize the cost $J(x,a)$ over all controls $a:\Omega\times[0,+\infty[\to+\infty$, progressively measurable.

We notice that $W_t^a = W_t - \int_0^t \sigma^{-1}R(a_s)\mathrm{d}s$ is a Brownian motion on $[0,T]$ under $\mathbb{P}_T^a$ and that

$$\mathrm{d}X_t^x = (b(X_t^x)+R(a_t))\mathrm{d}t + \sigma\mathrm{d}W_t^a, \quad \forall t\in[0,T],$$

and this justifies our formulation of the control problem.

We want to show how our results can be applied to such an optimization problem to get an asymptotic expansion of the finite horizon cost involving the ergodic cost.

To apply our results, we first define the Hamiltonian in the usual way,

$$f_0(x,z) = \inf_{a\in U}\left\{L(x,a)+z\sigma^1R(a)\right\}, \tag{5.27}$$

and we note that , if for all x,z the infimum is attained in (5.27), then by the Filippov theorem (see [59]), there exists a measurable function $\gamma:\overline{G}\times\mathbb{R}^{1\times d}$ such that

$$f_0(x,z) = L(x,\gamma(x,z)) + z\sigma^{-1}R(\gamma(x,z)).$$

Lemma 5.23. *Under the above assumptions, the Hamiltonian f_0 satisfies assumptions on f in Hypotheses 5.4, 5.5, 5.7, or 5.8.*

Proof. See Lemma 5.2 in [32]. □

We recall the following results about the finite horizon cost:

Lemma 5.24. *Assume that Hypotheses 5.1, 5.3, 5.6 and 5.9 hold true. Then for arbitrary control $a^T:\Omega\times[0,T]\to U$,*

$$J^T(x,a^T)\geq u(T,x),$$

where $u(t,x)$ is the viscosity solution of

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = \mathscr{L}u(t,x) + f_0(x,\nabla u(t,x)G), & \forall (t,x)\in\mathbb{R}_+\times G,\\ \frac{\partial u(t,x)}{\partial n} + g(x) = 0, & \forall (t,x)\in\mathbb{R}_+\times\partial G,\\ u(0,x) = h_0(x), & \forall x\in G,\end{cases}$$

Furthermore, if $\forall x$, z the infimum is attained in (5.27) then we have the equality:

$$J^T(x,\overline{a}^T) = u(T,x),$$

where $\overline{a}_t^T = \gamma(X_t^x,\nabla u(t,X_t^x)\sigma)$.

Proof. The proof of this result is similar to the proof of Theorem 7.1 in [31], so we omit it. □

Similarly, for the ergodic cost we have the following result.

Lemma 5.25. *Assume that Hypotheses 5.2, 5.3, 5.6 and 5.9 hold true, then for arbitrary control* $a : \Omega \times [0, +\infty[\to U$,

$$J(x, a) \geq \lambda,$$

where (v, λ) *is the viscosity solution of*

$$\begin{cases} \mathscr{L}v + f_0(x, \nabla v(x)\sigma) - \lambda = 0, & \forall x \in G, \\ \frac{\partial v(t,x)}{\partial n} + g(x) = 0, & \forall x \in \partial G. \end{cases}$$

Furthermore, if $\forall x$, z *the infimum is attained in* (5.27) *then we have the equality:*

$$J^T(x, \overline{a}) = \lambda,$$

where $\overline{a}_t = \gamma(X_t^x, \nabla v(X_t^x)\sigma)$.

Finally, we apply our result to obtain the following theorem.

Theorem 5.26. *Assume that Hypotheses 5.2, 5.3, 5.6 and 5.9 hold true. Then, for any control* $a : \Omega \times [0, T] \to U$, *we have*

$$\liminf_{T \to +\infty} \frac{J^T(x, a^T)}{T} \geq \lambda.$$

Furthermore, if $\forall x$, z *the infimum is attained in* (5.27) *then*

$$|J^T(x, \overline{a}^T) - J(x, \overline{a})T - v(x) + L| \leq Ce^{-\hat{\eta}T}.$$

Proof. The proof is a straightforward consequence of the two previous lemmas above and of Theorem 5.21. □

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Résumé

Équations différentielles stochastiques rétrogrades ergodiques

Cette thèse s'intéresse à l'étude des EDSR ergodiques et leur applications à l'étude du comportement en temps long des solutions d'EDP paraboliques semi-linéaires. Dans un premier temps, nous établissons des résultats d'existence et d'unicité d'une EDSR ergodique avec conditions de Neumann au bord dans un convexe non borné et dans un environnement faiblement dissipatif. Nous étudions ensuite leur lien avec les EDP avec conditions de Neumann au bord et nous donnons un exemple d'application à un problème de contrôle optimal stochastique.

La deuxième partie est constituée de deux sous-parties. Tout d'abord, nous étudions le comportement en temps long des solutions mild d'une EDP parabolique semi-linéaire en dimension infinie par des méthodes probabilistes. Cette méthode probabiliste repose sur une application d'un résultat nommé "Basic coupling estimate" qui nous permet d'obtenir une vitesse de convergence exponentielle de la solution vers son asymptote. Au passage notons que cette asymptote est entièrement déterminée par la solution de l'EDP ergodique semi-linéaire associée à l'EDP parabolique semi-linéaire initiale. Puis, nous adaptons cette méthode à l'étude du comportement en temps long des solutions de viscosité d'une EDP parabolique semi-linéaire avec condition de Neumann au bord dans un convexe borné en dimension finie. Par des méthodes de régularisation et de pénalisation des coefficients et en utilisant un résultat de stabilité pour les EDSR, nous obtenons des résultats analogues à ceux obtenus dans le contexte mild, avec notamment une vitesse exponentielle de convergence de la solution vers son asymptote.

Mots clefs : équations différentielles stochastiques rétrogrades ergodiques, contrôle ergodique, comportement en temps long, EDP parabolique semi-linéaires

Abstract

Ergodic backward stochastic differential equations

This thesis deals with the study of ergodic BSDE and their applications to the study of the large time behaviour of solutions to semilinear parabolic PDE. In a first time, we establish some existence and uniqueness results to an ergodic BSDE with Neumann boundary conditions in an unbounded convex set in a weakly dissipative environment. Then we study their link with PDE with Neumann boundary condition and we give an application to an ergodic stochastic control problem.

The second part consists of two sections. In the first one, we study the large time bahaviour of mild solutions to semilinear parabolic PDE in infinite dimension by a probabilistic method. This probabilistic method relies on a Basic coupling estimate result which gives us an exponential rate of convergence of the solution toward its asymptote. Let us mention that that this asymptote is fully determined by the solution of the ergodic semilinear PDE associated to the parabolic semilinear PDE. Then, we adapt this method to the sudy of the large time behaviour of viscosity solutions of semilinear parabolic PDE with Neumann boundary condition in a convex and bounded set in finite dimension. By regularization and penalization procedures, we obtain similar results as those obtained in the mild context, especially with an exponential rate of convergence for the solution toward its asymptote.

Keywords : ergodic backward stochastic differential equation, ergodic control, large time behaviour, semilinear parabolic PDE